The Teaching of

Secondary Mathematics

INSTITUTE OF EDUCATION FOR WOMEN
Dept. of Extension
SERVICE
CALCUTTA-27

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Secondary Mathematics

HARPER'S SERIES ON TEACHING

Under the Editorship of

ERNEST E. BAYLES

INSTITUTE OF EDUCATION FOR WOMEN
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The Teaching of Secondary Mathematics

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EDITOR'S FOREWORD

We are proud to introduce the Harper's Series on Teaching by way of a book which is written by a new author who gives evidence of high capability. Dr. Brown understands both mathematics and teaching theory. He knows his subject matter and he knows how to teach it; he has made serious study of both, and he presents the results of his study in a book which goes into the details of how to teach mathematics as well as why to do it that way.

This is no "neutral" book, neutral in the sense of having "no brief for any particular method." Certain ways of teaching seem better for us than do others because they embody the best thought on how children learn and because they square with the requirements of democratic education. Dr. Brown carefully considers these matters and makes the recommendations which appear to be required. His is a book which merits and will receive serious and respectful attention. We are grateful for the opportunity to present it to the profession.

Ernest E. Bayles

PREFACE

THE conflict of opinion regarding the nature of mathematics and its place in education has led to so much confusion and controversy that the average teacher of secondary mathematics is likely to be somewhat bewildered when he is faced with the problem of determining what mathematical subject matter to teach and how to teach it. Professional mathematicians tend to be impatient with these who view education as a science and to argue that anyone who has a broad knowledge of the subject matter of mathematics can teach it acceptably. On the other hand, although educational theorists have made extensive use of mathematics, especially of statistics, in their attempts to create a science of education, they have used it almost entirely for purposes of enumeration and measurement. For this reason, they commonly assume that counting and measurement constitute the whole field and function of mathematics. On the contrary, mathematicians view mathematics as primarily a mode of thinking—as the science which draws necessary conclusions—and insist that number and measurement constitute only one phase of the subject, and that immediate usefulness is not an adequate criterion of its value.

Obviously, there is little chance for agreement or common understanding when even the basic terms used by one group in discussing the problem mean something else to the other; when one group is not aware of or considers as of minor significance problems which loom large in importance to the other group. Now education is both a very old art and a very new science, and mathematics is both the science of number and measurement and something more. Hence the truth lies in neither the one viewpoint nor the other.

Preface

Part I of this book is an attempt to provide, in as concise form as possible, first, an account of the historical development of the present-day curriculum in secondary mathematics and of the historical origins of the conflict between the practical and the theoretical as it applies to mathematics and to mathematics teaching; second, an analysis of certain proposed teaching plans in so far as they have to do with mathematics education; and third, a teaching program which appears, in the light of evidence available, to promise resolution of the conflict of opinion over the place and function of mathematics in secondary education. Part II presents a discussion of the objectives and subject-matter content of specific courses in secondary mathematics together with some suggestions as to ways of solving some of the problems of teaching those subjects. Part III deals with general problems of classroom management, evaluation and marking as applied to courses in mathematics, and the preparation of competent teachers of secondary mathematics.

This book is based on the assumption that teaching method is individual; that a teacher who has an adequate knowledge of the subject matter of his field, who knows the needs, aspirations, and potentialities of his students, who understands the nature of the learning process, and whose philosophy of education is appropriate to a democratic society will be able to devise methods and techniques suited to his particular situation. The function of a book on the teaching of mathematics is, therefore, to provide an analysis of educational theory as applied to the teaching of secondary mathematics; to present the considerations which should govern choice of subject matter and of method; and to suggest possible methods of solving typical teaching problems.

Much of the content of a book like this is, of course, derived from the ideas and writings of others. While I have tried to indicate the sources of direct borrowings by footnotes, it is impossible to assess the full extent of my debt to others. However, I wish to make specific acknowledgment of my debt to the National Council of Teachers of Mathematics for permission to reprint materials from the Fifteenth

Yearbook, *The Place of Mathematics in Secondary Education,* and to the publishers of *School Science and Mathematics* and to Professors C. B. Read and G. R. Thacker for permission to reprint material from the article "Courses Desirable for Training Teachers of High School Mathematics." Many of the ideas presented in Chapter 5 were first published in *Educational Administration and Supervision,* Vol. XXVII, No. 6 (September, 1941), pp. 427–438, under the title "The Function of Mathematics in a Program of Education for Democratic Living," and are used here with the express permission of Warwick and York, Inc., Publishers. The material in Chapter 12 is in part taken from a report made to the Chicago Teachers College Conference by a committee composed of Dr. H. Glenn Ayre of the Western Illinois State College, Dr. Eugene Hellmich of the Northern Illinois State Teachers College, Dr. Lawrence A. Ringenberg of the Eastern Illinois State College, Dr. H. Van Engen of the Iowa State Teachers College, and myself.

I wish also to acknowledge my great debt to Dr. Ernest E. Bayles of the University of Kansas for his suggestions, encouragement, and constructive criticisms.

CLAUDE H. BROWN

Warrensburg, Missouri
February, 1953

PART I

Nature and Role of Secondary-School Mathematics

Chapter 1

Why Teach Mathematics?

THE number of students enrolled in the secondary schools of the United States has doubled every decade during the first half of the present century. In 1900, approximately one in ten of the youth of secondary-school age was enrolled in a secondary school. Now, approximately seven out of ten young Americans of high-school age are enrolled in secondary schools. These simple statistical facts are the basic causes for the controversy, unrest, and confusion that have characterized American education during the past half-century.

In 1900 the high school was a selective institution. The student body was made up largely of those who expected to enter college and those who would fill the upper-level white-collar jobs of the community and who, therefore, had the academic aptitudes and interests needed to succeed in the traditional, college-preparatory type of program offered. The student who could not succeed merely dropped out of school. The high schools of that day considered that a large number of failing students merely indicated that high scholastic standards were being maintained.

But, as the impact of increasing mechanization of industry and of agriculture was translated into a decreased demand for the services of youths of high-school age and as a generally rising standard of living made it possible for more and more young Americans of high-school age to attend high school, there arose increasing agitation for a more realistic curriculum. In particular, there was much sentiment for more "practical" courses such as the commercial subjects, industrial arts, vocational agriculture, and home economics. Since the school day was limited in length, these new courses could be added only by reducing the time devoted to the older, established subjects. Again, a considerable proportion of these newer students were not particularly interested in the old academic subjects. Many lacked the aptitude to profit from them. So educational theorists reexamined the basic philosophies of education, and practical administrators sought solutions to apparently insoluble problems of curriculum and staff in extensive curricular changes.

MATHEMATICS AS A TOOL SUBJECT

Since mathematics, not without reason, was considered one of the more difficult subjects, many reform proposals called for the virtual elimination of mathematics from secondary-school curriculums. Thus Snedden characterized algebra as "educational sawdust and gravel for second quarter intelligences" and argued that not more than 10 percent of secondary-school pupils should be permitted to take algebra and that these should take it for pre-vocational reasons only.[1] Wrinkle argued that "The outcomes of instruction in mathematics are primarily tool or service skills which are valuable only to the extent that they are usable."[2] Wrinkle also maintained that algebra and geometry were kept in the traditional secondary-school curriculum only because of college-entrance requirements. Counts thought that algebra and geometry contained but little material that was related to either present or probable needs of pupils.[3] Bobbitt wrote that "algebra, geometry, and trigonometry have a justifiable place in

[1] Quoted by William Betz in "Whither Algebra," *The Mathematics Teacher*, 23:106–107 (February, 1930).

[2] William L. Wrinkle, *The New High School in the Making*, New York, American Book Co., 1938, p. 180.

[3] Geo. S. Counts, "The Senior High School Curriculum," *Monographs on Education*, Chicago, University of Chicago Press, 1926, pp. 58–59. Quoted by William Betz, *op. cit.*, p. 108.

the curriculum only when they are necessary portions of vocational courses."[4] He admitted that applied mathematics was of value but questioned the educative worth of abstract phases of the subject. Prunty pointed with pride to the course of study developed at Tulsa, Oklahoma, from which units with "no practical applications in everyday life" had been eliminated,[5] whereas Finney insisted that the "rank and file of the people have use for only the elementary forms of computation"[6] and that "for putting our young democracies to death without blood-letting or commotion, Latin, rhetoric, mathematics, and foreign languages would have proved the most effective brand of chloroform that could have been devised."[7]

The foregoing should suffice to indicate one line of thought about the place of mathematics in secondary education—that, judged by the criterion of utility, mathematics lacks sufficient immediate applicability to warrant inclusion in the secondary-school curriculum.

MATHEMATICS AS CULTURAL

The opposite opinion as to the educative value of mathematics was expressed by Keyser:

> No, the belief that mathematics, because it is abstract, because it is static and cold and grey, is detached from life, is a mistaken belief. Mathematics, even in its purest and most abstract estate, is not detached from life. It is just the ideal handling of the problems of life, as sculpture may idealize a human figure or as poetry or painting may idealize a figure or a scene. Mathematics is precisely the ideal handling of the problems of life, and the central ideas of the science, the great concepts about which its stately doctrines have been built up are precisely the chief ideas with which life must always deal and which, as it tumbles and rolls about

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[4] Franklin Bobbitt, "Curriculum Making in Los Angeles," *Monographs on Education,* Chicago, University of Chicago Press, 1922. Quoted by William Betz, *op. cit.,* p. 108.

[5] Merle Prunty, "The New Curriculum," *School Executives Magazine,* 54: 3–5 (September, 1934).

[6] Ross L. Finney, *A Sociological Philosophy of Education,* New York, The Macmillan Company, 1928, p. 174.

[7] *Ibid.,* p. 286.

them through time and space, give it its interests and problems, and its order and rationality.[8]

Keyser contended that all knowledge, and mathematics in particular, needed no utilitarian justification; it was an end in itself.[9] To him mathematics was cultural, therefore useful. Altschiller-Court contended that mathematics is useful both in the sense of being cultural, as Keyser maintained, and also in the sense that art, music, and literature are useful.[10] Obviously, the term *useful* as applied to mathematics either did not mean the same thing to Keyser and Altschiller-Court that it did to Wrinkle, Snedden, Bobbitt, and the other writers previously quoted or they had in mind different groups to whom mathematics was to be useful.

Atherton probably had a conception of the term *useful* as applied to mathematics that was wider than the conception implied by the criterion of immediate applicability when he pointed out that progress in scientific research is limited chiefly by the limitations of our knowledge of mathematics.[11] Hart pointed out the growing demand for mathematics as a relatively new feature in many fields not formerly considered to have mathematical interests.[12] He insisted that all secondary mathematics by itself was both culturally and practically useful, regardless of later college contacts.

Bell also apparently had this wider concept of usefulness in mind when he pointed out how time after time major scientific theories have come into being only because the very ideas in terms of which these theories had meaning were created by mathematicians years, decades, or even centuries before anyone foresaw possible applications to science.[13] He mentioned as an example the dependence of Einstein's theory of relativity upon the geometry of Riemann, developed half a century before the time of Einstein. Also, much of the theory of the functions of a complex variable was developed long before it was applied to electrical theory. Bell went so far as to say that in mathematics the attempt to create something of immediate utility leads to shoddy work of little or only passing value as a rule, whereas creative mathematicians are inspired by the *art* of mathematics rather than by any prospect of ultimate usefulness.

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[8] Cassius J. Keyser, *The Human Worth of Rigorous Thinking*, New York, Columbia University Press, 1916, p. 77.

[9] *Ibid.*, pp. 183–184.

[10] Nathan Altschiller-Court, "Art and Mathematics," *Scripta Mathematica*, 3:103–111 (April, 1935).

[11] Charles R. Atherton, "The Place of Mathematics in the Curriculum of the Progressive School," *The Mathematics Teacher*, 24:48–55 (January, 1935).

[12] W. L. Hart, "The Need for a Reorientation of Secondary Mathematics from a College Viewpoint," *The Mathematics Teacher*, 28:69–79 (February, 1935).

CONFLICT BETWEEN THE PRACTICAL AND THE CULTURAL

In 1938 the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics published a preliminary report on "The Place of Mathematics in Secondary Education." In it was this statement about the usefulness of mathematics:

> Inasmuch as mathematics grew out of needs we should expect it to be useful. But as the subject expanded it became more and more self-contained, mathematicians taking delight with the subject itself, regarded it as one of man's most appropriate activities. Such a change of attitude has given an even greater incentive for pressing its development than the mere goad of need, but, nevertheless, contacts with the physical world and applications to it have been discovered as by-products, such results being constantly taken up and cultivated by those engaged in various technical pursuits. As there is no danger of being deprived of the results of mathematical inquiries, people grow unappreciative, failing to recognize the many ways in which the subject ministers to us and forgetting how primitive civilization would be without it. There are probably people, for example, who do not know that there is quite as much use made of differential equations as of wire in a radio set. An inspection of texts having to do with numerous technical fields, professions, and trades reveals their great dependence upon mathematics, and such an inspection made thoughtfully can well be recommended for those who set themselves up as competent appraisers of our culture.[14]

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[13] Eric T. Bell, *The Queen of the Sciences*, Baltimore, The Williams and Wilkins Co., 1931, chap. 1.

[14] A Preliminary Report of the Joint Commission of the Mathematical Associa-

The preliminary report went on to mention newer uses of mathematics, such as the greater mathematizing of physics, and the extension of mathematics to fields which, until recently, were not mathematical at all. The increasing use of mathematics in such fields, of which psychology may be taken as an example, was cited as evidence that the average person is likely to need more rather than less mathematical training in the future.

It is readily apparent that there is wide divergence in the interpretation of the term *useful* as applied to mathematics. Bell tells a story about Fourier and Jacobi which illustrates rather well the two interpretations of the word. It seems that Fourier had reproached Abel and Jacobi for "wasting their time on elliptic functions while there were still problems in heat conduction to be solved." (Useful, practical problems, if you please.) Bell quotes Jacobi's answer as follows: "It is true that M. Fourier had the opinion that the principal aim of mathematics was public utility and the explanation of natural phenomena, but a philosopher like him should have known that the sole end of science is the honor of the human mind, and that under this title a question about numbers is worth as much as a question about the system of the world."[15]

Obviously, the group who argued that mathematics has only limited usefulness, hence that curricular offerings in it should be strictly limited, and the group who argued for the almost universal need for mathematics could not reach agreement unless some means could be found to insure that they meant the same thing when they used certain terms. But the dispute was more fundamental than mere quibbling over a word and its meaning. In part, the controversy pertained

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tion of America and of the National Council of Teachers of Mathematics, *The Place of Mathematics in Secondary Education*, Ann Arbor, Michigan, Edwards Bros., Inc., 1938, pp. 29–30.

(The particular passage quoted here does not appear in the final report of the Joint Commission, published as the Fifteenth Yearbook of the National Council of Teachers of Mathematics, Bureau of Publications, Teachers College, Columbia University, 1940.)

[15] Eric T. Bell, *Men of Mathematics*, New York, Simon and Schuster, Inc., 1937, p. 338.

to groups to be educated, not to subject matter alone. What was good for the few might not be appropriate for the many. Snedden, Wrinkle, Bobbitt, and the other educators quoted on pages 4–5 did not deny the value of mathematics for capable students, but they did question the wisdom of making it a required subject for all. On the other hand, Keyser and the other mathematicians quoted on pages 5–8 nowhere claimed that the potential values inherent in the study of mathematics could be achieved by low-ability students.

One explanation of the widely divergent views which have been presented concerning the educative values of mathematics is that of Keyser, who said:

> Nowhere is the confusion of the time more evident than in the somewhat noisy and sometimes acrimonious discussion that has been recently and still is going on throughout our country regarding the value of mathematics as a subject in secondary and collegiate education. The instigators of the discussion, those, that is, who advocate so reducing mathematical requirements as practically to abolish the subject from the curricula of general education, are not malicious or insincere; many of them, I do not doubt, are well-meaning citizens. And if their rather voluminous discourses are often singularly lacking in coherence, in clarity, and in depth, the defects are not due to evil intentions but rather, I suspect, to confusion and a lack of just that sort of discipline which the subject the authors are engaged in depreciating is peculiarly qualified to give. Perhaps we should not be astonished. If the saying of Sir Oliver Lodge be true that "the mathematical ignorance of the average educated person has always been complete and shameless," one ought not, I suppose, be too much astonished if in a vast, crude, formless, sprawling democracy like ours, a way to educational "leadership" is sometimes found by men whose innocence, not only of mathematics but of the other great subjects, including the principles of education, is well-nigh complete and shameless.[16]

A less partisan explanation is that the conflict over the place of mathematics in secondary education had its origin in differing conceptions as to the nature of mathematics and as to the aim and function of education.

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[16] Cassius J. Keyser, *Mathematical Philosophy*, New York, E. P. Dutton and Co., 1922, p. 13.

THE NATURE OF MATHEMATICS

As Sanford points out, on the one hand mathematics deals "with quantitative relationships between material objects and thus it becomes a tool in the world of business economics or science," whereas, on the other hand, "it deduces theorems from arbitrarily chosen postulates and seeks to carry these theorems to their logical conclusions."[17] This second phase of mathematics gives but little heed to the immediate utility of its theorems and conclusions.

It is from these two aspects of the subject that the differing definitions of mathematics spring. It appears to be the first phase that Schorling and Clark have in mind, for example, when they define mathematics as "the science of number and quantity."[18] This is essentially the view implied in Webster's *Dictionary*, which defines mathematics as "That science or class of sciences which treats of the exact relations existing between quantities or magnitudes and operations and of the methods by which, in accordance with these relations, quantities sought are deductible from others known or supposed; the science of serial, spatial, quantitative and magnitudinal relations; the science of order."

Such definitions carry the implication of something tangible to be measured. That quantities, magnitudes, and the relations between them are things evident to the senses—tangibles that we can count—is implied in the conception of mathematics as the science of order. Obviously, if mathematics applied to the relations among tangible things, if it has to do merely with counting and measuring, then the only usefulness it could have would be limited to counting and measuring. The objections to the supposed cultural and disciplinary values of such a subject would then be reasonable.

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[17] Vera Sanford, *A Short History of Mathematics*, Boston, Houghton Mifflin Co., 1930, pp. 1–2.

[18] Raleigh Schorling and John R. Clark, *Mathematics in Life*, Yonkers, New York, World Book Co., 1937, p. 11.

But there are other conceptions, other definitions of mathematics. For example, Bell quotes Edward Everett as follows: "In the pure mathematics we contemplate absolute truths, which existed in the divine mind before the morning stars sang together, and which will continue to exist there when the last of that radiant host shall have fallen from heaven."[19] Here we have the conception of mathematics as eternal truth. This is the conception that was held by Plato; to the ancient Greeks mathematics represented absolute truth. If such is to be the definition, quite obviously it does not matter whether mathematics is immediately applicable. In fact, the Greeks regarded absolute or ultimate truth with something akin to religious awe and, believing that the conclusions of theoretical mathematics were absolute truths, deliberately refrained from making practical applications of them.

To idealists there is a fundamental dualism inherent in mathematics. When mathematical concepts are applied to tangible things the conclusions reached may be verified empirically. But conclusions concerning the relationships between idealized concepts cannot be so verified. The relations between tangible things belong to the material world and are therefore approximate and subject to error and change, whereas relations between idealized concepts belong to the ideal world and are therefore exact, unchanging, and eternal. The Greeks chose to study these ideal relationships—absolute truths—and to ignore empirical relationships.

Bell believes that a satisfactory statement of the nature of mathematics will place major emphasis on the abstract phase. He quotes Bertrand Russell as follows: "Mathematics may be defined as the subject in which we never know what we are talking about nor whether what we are saying is true."[20] Bell likes Russell's definition because it emphasizes the abstract character of mathematics and effectually minimizes the idea of mathematics as the science of number, quantity, and measurement. He says that these things "bear about the same relation to mathematics that oil and ground ochre bear to great art."

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[19] Eric T. Bell, *The Queen of the Sciences,* p. 20.

[20] *Ibid.,* p. 16.

Keyser expresses the same idea thus: "Mathematics is no more the art of reckoning and computation than architecture is the art of making bricks or hewing wood, no more than painting is the art of mixing colors on a palette, no more than the science of geology is the art of breaking rocks, or the science of anatomy is the art of butchering."[21] Keyser believes that the conception of the nature of the science of mathematics has grown with the science itself. For example, he says:

> A traditional conception, still current everywhere except in critical circles, has held mathematics to be the science of quantity or magnitude, including multitude (with its correlate of number) as a special kind, signified whatever was "capable of increase and decrease and measurement." Measurability was the essential thing. That definition of the science was a very natural one for magnitude did appear to be a very fundamental notion, not only inviting but demanding consideration at every stage and turn of life. The necessity of finding out how many and how much was the mother of counting and measurement, and mathematics, first from necessity and then from pure curiosity and joy, so occupied itself with these things that they came to seem its whole employment.[22]

Keyser shows that the ordinary concept of measurement as the repeated application of a constant finite unit does not apply to much of modern mathematics; in fact Comte defined mathematics as "the science of indirect measurement." The most striking of modern measurements, such as cell growth, astronomical measurements, rates of chemical change, or valence of atoms, are accomplished by indirection. More than that, much of modern mathematics has nothing to do with any sort of measurement. The whole field of projective geometry has little or no concern with quantity or magnitude, and metric considerations seldom enter into the conclusions. Its theorems deal with those properties of lines, points, and planes which are independent of lengths of line segments or angle magnitudes.

[21] Keyser, *The Human Worth of Rigorous Thinking*, p. 298.
[22] *Ibid.*, pp. 276 ff.

We may grant all that the defenders of mathematics have said about its value to civilization, however, and still doubt the wisdom of making it a required subject for all secondary-school pupils. Similarly, we may admit that algebra and geometry possess but little immediate vocational utility and yet feel that mathematics has a definite place in the secondary-school curriculum.

Hence, it appears that the dispute concerning the place of mathematics in secondary education is due, at least in part, to the dual nature of mathematical interpretation which makes it difficult to formulate a satisfactory definition for the subject. A definition which places major emphasis on the practical phase gives a distorted view of the subject if it implies that quantitative relationships between tangible things comprise the whole of mathematics. On the other hand, definitions which place the emphasis on the theoretical phase frequently fail to differentiate between mathematics and logic.

Arguments for reducing the extent of curricular offerings in mathematics are usually based on definitions which stress the practical and utilitarian aspects of the subject. Definitions which imply that the conclusions of abstract mathematics are absolute truths—eternal verities—stress the precision and permanence of the conclusions reached by mathematical processes. The group who would teach mathematics because it develops in a student the ability to reason base their arguments on the assumption that mathematics is a postulational science. They contend that it is first and foremost a mode of thinking. The possible practical applications of its principles and the relative certainty and permanence which may be ascribed to its conclusions are secondary, not primary, values to be gained from the study of mathematics.

Obviously the real point at issue is not whether mathematics should be included in the secondary-school curriculum but rather what, and how much, mathematics should be included, and for whom. It is manifestly impossible for any individual, however gifted, to master the whole field. Hence it is evident that any program of mathematics education for all students implies a selection of subject matter; or a prescription of subject matter, if it is to be educationally defensible, implies a selection of students.

But selection of subject matter and of students is the point at issue in the dispute as to whether education should be practical or cultural. A satisfactory solution of this problem in so far as it pertains to mathematics must give proper emphasis to both phases of the subject since both make real and unique contributions to the education of an individual.

MATHEMATICS AS BOTH PRACTICAL AND CULTURAL

The dispute between those who would largely eliminate mathematics from the curriculum and those who regard a knowledge of mathematics as essential to a liberal education is not of recent origin. Heath quotes the Greek Sophist, Isocrates, as having said that studies such as astronomy and geometry, even if they do no other good, keep the young out of mischief, and that in his opinion no other subjects could have been invented more useful and more fitting.[23] Most people, according to Isocrates, are likely to think astronomy and geometry idle since they are of no use in public or private affairs; moreover, they are quickly forgotten because they do not go with us into our daily life and action but are altogether outside our daily needs. Isocrates admits that those who specialize in subjects such as astronomy and geometry get no good from them unless they choose to teach them for a livelihood. He further admits that if a student becomes too deeply absorbed in these subjects he may become impractical and incapable of doing ordinary business. Isocrates maintains, however, that the study of these subjects up to the proper point trains a boy to keep his attention fixed and not to allow his mind to wander. Hence, as a result of having his wits sharpened and of being practiced in this way by the study of mathematics, he thereby becomes capable of learning more important matters with greater ease and speed.

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[23] Sir Thomas L. Heath, *A History of Greek Mathematics*, London, The Clarendon Press, 1921, Vol. I, p. 21.

Isocrates was apparently trying to meet arguments similar in tenor to those now urged in favor of eliminating much of mathematics from the curriculum. Evidently the dispute as to the educative value of mathematics cannot be an outgrowth of the overthrow of the doctrine of formal discipline since it antedates considerably the formulation of that doctrine. In fact, the dispute is comparatively recent only in so far as it concerns schooling for all youth of secondary-school age. The division of opinion over the place of mathematics in secondary education seems to be a variation of the dispute over the relative merits of pure and applied science, which goes far back into the history of human thought.

In mathematics the conflict between the practical and the theoretical is probably brought into sharper focus than in other fields. It is comparatively easy to determine just what simple arithmetical skills the average person (whatever that term may mean) uses. Tables and calculating machines and other devices are available which eliminate much of the computation once necessary. These facts may be cited as evidence of the nonfunctional character of secondary-school mathematics. On the other hand, it is equally easy to point out the infinite applicability of mathematical theory to all fields of human endeavor and, as a consequence of this, to claim that teaching of the most abstract mathematics is justified because it is useful.

The essential idea behind the practical- or applied-science point of view is that the value of a thing is determined by the use that may be made of it. Hence, a theory in mathematics is worthwhile only as it aids in solving specific practical problems. A study is of value in education only as it serves the specific and immediately felt needs of a pupil.

From the pure-science point of view, a new theory in mathematics is of value in and of itself. It is a work of art, like a painting, a poem, or a musical composition. If a practical use be found for this theory, either at the time or later, the added value is only incidental, as is the value added to the pleasure of owning a beautiful tapestry because it also serves to cover a stained spot on the wall. A school subject is valuable if it merely serves to give a glimpse of the cultural heritage of the past. According to this view, Aristotle is still worthy of study, not as a source of information—the original reason for studying his work—but as a part of the cultural background of our civilization.

Neither view gives a sufficiently broad outlook on life to permit complete exclusion of the other. The problem of determining the place of mathematics in secondary education is precisely that of striking a proper balance between practical and theoretical phases of the subject in education.

The attention of theoretical mathematicians has been too much centered on attempts to justify the subject because of its intrinsic value. Practical educators, in attempting to meet the needs of the heterogeneous groups that fill classrooms, tend to be more concerned with immediate utility than with more remote and less tangible benefits. No one doubts that this is a mathematical civilization or that there is value in mathematical research. Nor does anyone doubt that a great many students fail to profit from the mathematical courses now offered in the schools. The "forgotten man" in mathematical education is the student who is not interested in becoming a research worker but who expects to gain from the study of mathematics values other than the purely practical ones of being able to do the relatively simple computations needed in everyday life.

The concept that mathematics is a means of communicating ideas, as suggested by Black,[24] by Hogben,[25] and by Dantzig,[26] seems to avoid undue emphasis on either phase of the subject. It is comparatively easy to see the close relationship between the idea of mathematics as the science of number, quantity, and measurement and the idea of mathematics as a language for the expression of quantitative ideas. The group who want to cut down the mathematical offerings of the curriculum will probably agree that it is desirable for an individual to learn enough of the language of numbers to express his quantitative ideas with ease and precision.

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[24] Max Black, *The Nature of Mathematics*, New York, Harcourt, Brace and Co., 1933.

[25] Lancelot Hogben, *Mathematics for the Million*, New York, W. W. Norton and Co., 1937.

[26] Tobias Dantzig, *Number, The Language of Science*, New York, The Macmillan Company, 1930.

The means by which the group who regard mathematics as a mode of thinking—who stress the theoretical phase—may be reconciled with the idea of mathematics as a language is, perhaps, less immediately evident. These persons object to the idea of mathematics as merely a tool—a means of obtaining an answer. However, the idea of mathematics as a language—as a tool for thinking—implies a broader view of its usefulness than the generally accepted concept of a tool subject.

Briefly, this newer concept may be stated as follows: Words are the symbols with which we express ideas in literary language; numbers are the symbols used in mathematical language. Symbols are so great an aid to thinking that it is difficult to conceive of thought except in terms of symbols. Mathematics is a symbolic language whose sole function and purpose is to facilitate thinking. It was first developed in connection with quantitative relations but is not necessarily confined to the expression of such relations. Mathematics is characterized by the precision which it imparts to thinking and by the objective nature of the conclusions reached.

The two phases of mathematics, that of being a language for communicating or identifying quantitative ideas and serving as a tool for thinking and that of expressing quantitative relations between tangible things, are illustrated by Brown in his paper on the applications of mathematics to psychology.[27] He mentions that one use of mathematics in physics is measurement of physical phenomena, but that an equally important application is the use of mathematical theories as guides in the formulation and testing of hypotheses. In psychology, statistical measurement is used, but the more important use of mathematics in psychology is that of formulating hypotheses leading to discovery of new truths. The particular field of mathematics utilized in this manner in psychology is that of *topology*, which ostensibly makes no use of quantitative concepts at all.

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[27] J. F. Brown, "On the Use of Mathematics in Psychological Theory," *Psychometrika,* 1:77–90 (1936).

The concept of mathematics as a symbolic language which facilitates thinking does not do violence to the ideas of a technical mathematician. Research in pure mathematics is analogous to creative writing in literature in that it creates new forms of expression, new ways of thinking. Like the work of a creative artist, the value of abstract mathematics is not measured by its utility but by its artistic merit—that is, by the rigor and clarity of the demonstration. Educationally, the study of mathematics, so defined, surely affords training in clear thinking.

Hence, *mathematics may be regarded as a symbolic language whose practical function is to express quantitative and spatial relationships and whose theoretical function is to facilitate thinking.* This conception of mathematics recognizes and gives due emphasis to both phases of the subject. It may well serve as the basis for a satisfactory solution of the dispute concerning the place of mathematics in secondary education.

But agreement on a definition of mathematics, even if it be secured, will not settle the dispute over the function of mathematics in secondary education. There is still the problem of determining what, if any, are the educative values of mathematics, by whom they may be secured, and how best to realize them in practice.

Does mathematics have educative values not possessed by other subjects in the curriculum? Unless it can be shown that the contribution of mathematics to the development of an individual is to a degree unique, it is almost certain to be relegated to a position of secondary importance in the curriculum. The value of mathematical studies as preparation for certain vocations, such as the engineering profession, is not questioned. The desirability of providing opportunities for those who may wish to do research work in mathematics to obtain the requisite preliminary training is not denied. But that any considerable amount of mathematics should be required of all secondary-school students is disputed, and disputed vigorously.

The controversy over the place of mathematics in education appears to have its origin in the dispute over the relative utility of the practical and the cultural in education. We shall be concerned, therefore, with studying this conflict as it applies to mathematics and to mathematics teaching. In particular, since it is charged that mathematics owes its place in the high-school curriculum to tradition and to college-entrance requirements, we shall begin by investigating the origin and development of the mathematics curriculum as it exists in the public schools of the United States today.

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Chapter 2

Historical Background of Today's Curriculum

FULL understanding of the forces which have formed the present curriculum in mathematics requires knowledge of the history of mathematics itself. It also requires knowledge of the history of education since the school program is a function of both subject matter and prevailing philosophy of education. Since the basic mathematics subjects offered in the public schools of the United States at present are arithmetic, algebra, and geometry, or various composites thereof, this survey will be primarily concerned with these subjects.

ORIGIN OF MATHEMATICS IN PRACTICAL PROBLEMS

Mathematics is usually represented as a result of man's attempts to solve certain quantitative problems which have confronted him in his daily living. Counting is believed to have originated in an attempt to answer the question "How many?" This question would be sure to arise in recounting the results of an expedition after game or to make war on a neighboring tribe. An excellent idea of how primitive concepts of measurement may have sprung up can be obtained by considering some of the words used to convey ideas of quantitative relationships. Terms such as foot, pace, hand, rod, score, tally, and span, and folk expressions such as "count noses," are vivid evidence of the origin of measurement and units for measurement in the everyday life of the race.

It is significant that primitive man has always been a bit superstitious about number concepts. The results that can be obtained by manipulation of numbers seemed to savor of the magical. Hence we find distinctions between odd and even numbers. Primes were set off as having unusual powers, and numbers such as three, seven, and thirteen were invested with magic potency. Perhaps the first beginnings of the dispute as to advisability of studying mathematics for its practical values or for theoretical considerations may be found here.

As civilization developed, new uses were found for mathematics. As long as life was nomadic and pastoral, the enumeration of flocks and herds, of tribesmen, and of wives and children essentially encompassed an individual's need of number and mathematics. But when the way of life changed to that of a settled agricultural community, problems of measurement of land and grain, of determining seasons, and of carrying on business arose. Smith cites evidence of bills, notes, receipts, accounts, and systems of weights and measures in Mesopotamia as early as 3000 B.C.[1] Of particular interest is the Babylonian calendar and the use of sixty in expressing the relations between units of time and of angular measure. Our modern angular measure dates back to the Chaldeans.

In Egypt, the restricted amount of usable land gave problems of land survey vital importance.[2] The peculiar dependence of the farmer upon the Nile placed great weight upon the determination of seasons and the prediction of probable dates and heights of floods. Hogben says that the river records kept were the property of the priests, who made use of them to increase the awe and superstition with which the common man regarded number.[3]

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[1] David Eugene Smith, *History of Mathematics*, Boston, Ginn and Co., 1923, Vol. I, pp. 35–41.

[2] Sir Thomas L. Heath, *A History of Greek Mathematics*, London, The Clarendon Press, 1921, Vol. I, p. 120.

[3] Lancelot Hogben, *Mathematics for the Million*, New York, W. W. Norton and Co., 1937, pp. 36–68.

In Mexico the Mayas, another highly civilized agricultural people, likewise developed a number system and astronomy, and had a calendar that was far superior to that used in Europe at the time America was discovered.

As to the methods of teaching mathematics used by ancient peoples other than the Greeks, we know little except that teaching was probably almost entirely in the hands of the priests and the subject matter was studied principally because of its usefulness, real or fancied. Today we might not regard Babylonian and Chaldean astrology as having use value but there is no doubt that the people of that day regarded it as very useful.

Of some points, however, we can be reasonably sure. One is that in Egypt and Babylonia arithmetic and geometry developed to rather high degree. Cajori cites evidence that in these countries mathematics was developed to such a point that the everyday needs of the people were thoroughly met but that then it stagnated.[4] Sanford says that in Egypt both the new conditions that would have made new theory necessary and, failing that, the desire to study mathematics for its own sake were lacking.[5] Sanford also mentions that in China there was sufficient leisure for philosophic speculation and the desire to undertake it, but the results of Chinese studies were not available to the Occident until long afterwards; hence they had no effect on the development of Western mathematics.

THE GREEKS DEVELOPED DEMONSTRATIVE GEOMETRY FOR ITS CULTURAL VALUES

Among the peoples with whom the Egyptians carried on commerce were the Greeks. In consequence, Greece obtained not only the goods of the Egyptians but also their scientific ideas, particularly in astronomy and mathematics. Proclus says that one of the Greek merchants, Thales, became interested in astronomy and geometry and about 600 B.C. introduced geometry into Greece.[6] One story told of Thales is to the effect that one year, when he feared that a bountiful olive crop would unduly lower the price of the oil and thereby reduce his profits, he bought up all the available olive presses and thus secured a "corner on the market."[7] It is evident that Thales could deal with practical problems. His great contribution to the world, however, was not the technique of price manipulation, which has perhaps been used too much, but the idea of mathematical proof. According to Allman, "The only geometry known to the Egyptian priests was that of surfaces, together with a sketch of that of solids, a geometry consisting of some simple quadratures and elementary cubatures, which they had obtained empirically. Thales, on the other hand, introduced *abstract* geometry, the object of which is to establish precise *relations* between the different parts of a figure, so that some of them could be found by means of others in a manner strictly rigorous."[8] Allman also says that Thales laid the foundation of algebra and was the first to make practical application of theoretical geometry.[9]

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[4] Florian Cajori, *A History of Mathematics,* New York, The Macmillan Company, 1931, pp. 4–14.

[5] Vera Sanford, *A Short History of Mathematics,* Boston, Houghton Mifflin Co., 1930, p. 4.

Pythagoras, possibly a student of Thales, traveled in Egypt and may have studied in Babylonia and India.[10] He founded a school at Crotona, in Italy, much like a secret fraternal society and there expounded his theories about the nature of mathematics. According to Allman, "The Pythagoreans first severed geometry from the needs of practical life and treated it as a liberal science, giving definitions, and introducing the manner of proof which has ever since been in use."[11]

The Pythagoreans divided mathematics into four branches—

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[6] Heath, *op. cit.*, p. 128.

[7] W. B. Frankland, *The Story of Euclid*, London, Hodder and Stoughton, 1901, p. 24.

[8] George J. Allman, *Greek Geometry from Thales to Euclid*, London, Longmans, Green and Co., 1889, p. 7.

[9] *Ibid.*, pp. 16–17.

[10] Smith, *op. cit.*, Vol. I, pp. 71–72.

[11] Allman, *op. cit.*, p. 48.

geometry, astronomy, arithmetic, and music—a division that persisted until the Middle Ages.

Again quoting Allman:

> One chief characteristic of the mathematical work of Pythagoras was the combination of arithmetic with geometry. The notions of an equation and a proportion—which are common to both, and contain the first germ of algebra—were, as we have seen, introduced among the Greeks by Thales. These notions, especially the latter, were elaborated by Pythagoras and his school, so that they reached the rank of a true scientific method in their Theory of Proportion. To Pythagoras, then, is due the honor of having supplied a method which is common to all branches of mathematics, and in this respect he is fully comparable to Descartes, to whom we owe the decisive combination of algebra with geometry.[12]

By arithmetic the Greeks meant the science of number; they called the computational phase of arithmetic, for them largely the use of the abacus, logistic. Heath says that, although Plato did not ignore the art of calculation, he considered it "only preparatory to the true science; those who are to govern the city are to get a grasp of [the art of calculation], not in the popular sense with a view to use in trade, but only for the purpose of knowledge, until they are able to contemplate the nature of number by thought alone."[13]

It is significant that the period of three hundred years between Thales (c. 600 B.C.) and Euclid (c. 300 B.C.) gave rise to the whole of the theory of geometry regarded as purely abstract, whereas during that time computational arithmetic (logistic) made little or no progress. It was considered to be of only practical value and beneath the notice of those who sought intellectual discipline.

A sharp distinction between practical and theoretical was characteristic of Greek philosophy. Greek philosophers postulated the existence of an ideal world—the world of ideas—separate and apart from the material world—the world of things. The world of things was the concern of artisans and slaves; the world of ideas was for freemen only. Nicomachus said:

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[12] *Ibid.*, p. 48.

[13] Heath, *op. cit.*, Vol. I, p. 13.

> . . . And before Pythagoras all who had knowledge were called "wise" indiscriminately—a carpenter, for example, a cobbler, a helmsman, and in a word any one who was versed in any art or handicraft. Pythagoras, however, restricting the title so as to apply to the knowledge and comprehension of reality, and calling the knowledge of the truth in this the only wisdom, naturally designated the desire and pursuit of this knowledge philosophy, as being desire for wisdom. . . . This "Wisdom" he defined as the knowledge, or science, of the truth in real things, conceiving "science" to be a steadfast and firm apprehension of the underlying substance, and "real things" to be those which continue uniformly and the same in the universe. . . . These "real things" would be things immaterial.[14]

Nicomachus continued:

> For bodily, material things are, to be sure, forever involved in continuous flow and change—in imitation of the nature and peculiar quality of that eternal matter and substance which has been from the beginning, and which was all changeable and variable throughout. The bodiless things, however, of which we conceive in connection with or together with matter, such as qualities, quantities, configurations, largeness, smallness, equality, relations, actualities, dispositions, places, times, all those things, in a word, whereby the qualities found in each body are comprehended—all these are of themselves immovable and unchangeable, but accidentally they share in and partake of the affections of the body to which they belong. Now it is with such things that "wisdom" is particularly concerned, but accidentally also with things that share in them, that is, bodies.[15]

In other words, relationships in the material world were subject to change, hence did not represent ultimate truth. But relations in the ideal world were unchanging—were ultimate or absolute truth—and the proper concern of a philosopher.

Aside from the theorem that is named for him, Pythagoras is best known for his ideas concerning the nature of number. He ascribed mystic properties to number. For example, he regarded the odd numbers as male and the even numbers as female. The number

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[14] Nicomachus of Gerasa, *Introduction to Arithmetic*, transl. by Martin Luther D'Ooge, New York, The Macmillan Company, 1926, p. 181.

[15] *Ibid.*, p. 182.

"one" was identified with God the Creator because all others came from it. Seven had unusual powers "because of all the numbers of the decade it alone is both prime and does not as a factor produce any other number within the decade—neither generates or is generated, as the Pythagoreans said—it is called 'Athena,' the virgin goddess."[16]

Pythagoras believed that number constitutes the pattern of the universe. In the words of Nicomachus, "the devising Mind wrought with reference to the resemblances and likenesses of number, as to a pattern that was perfect, the fabric and composition of the world and of all that is in it."[17] Therefore number—mathematics—represents ultimate truth. To know about number is to know the mind of Deity; to think mathematically is to think as the gods think.

The Greeks, therefore, severed the connection between mathematics and practical things. Plato made geometry a prerequisite for entrance into his academy because he wished his students to have knowledge of the world—to be sophisticated. He esteemed geometry and arithmetic highly as means of compelling the mind to reason about abstract things. Plato laid the logical foundations of geometry by insisting upon clearly stated assumptions and definitions and on logical proofs. If not due to Plato, the restriction of geometry to figures such as can be drawn with straightedge and compass was certainly in keeping with Plato's philosophy of geometry. For, according to Plutarch, Plato vigorously condemned the practice, begun by Eudoxus and Archytas, of using certain mechanical aids to solve geometrical problems. Plutarch says:

> . . . To solve the problem, so often required in constructing geometrical figures, given the two extreme, to find the two mean lines of a proportion, both these mathematicians had recourse to the aid of instruments, adapting to their purpose certain curves and sections of lines. But what with Plato's indignation at it, and his invectives against it as the mere corruption and annihilation of the one good of geometry,—which was thus shamefully turning its back upon the unembodied objects of

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[16] *Ibid.*, p. 106.
[17] *Ibid.*, p. 98.

pure intelligence to recur to sensation, and to ask help (not to be obtained without base subservience and depravation) from matter; so it was that mechanics came to be separated from geometry, and repudiated and neglected by philosophers, took its place as a military art.[18]

The implications of Plato's criticism that the methods of Eudoxus and Archytas "recur to sensation" are made clear by the statement of Proclus that "everything that is in the course of becoming progresses from the imperfect to the perfect. Thus the transition from sensation to reasoning and from reasoning to understanding is only natural."[19] In other words, Plato regarded the action of Eudoxus and Archytas as a backward step.

The task of reducing geometry and all Greek mathematics to a postulational basis was the great work of Euclid. So far as is known, Euclid added nothing new to geometry. His great contribution was reduction of the mass of material already developed to a logical system with assumptions clearly stated. Euclid eliminated all practical applications from geometry, and his organization of the subject matter has persisted with relatively little change to the present time. In fact, Bell complains that modern high-school geometry is twenty-two hundred years old and that high-school algebra extends little further than man's thinking had progressed by 1650.[20]

Summarizing, we find that the direction of early development in geometry was from practical to theoretical. The ancients up to and including the Egyptians studied only the practical aspects of the subject. Rule-of-thumb approximations of formulas satisfied them and no attempt was made at proof of any proposition. Also, and quite significantly, as soon as the practical problems which demanded solution were solved, progress in the science ceased.

Although the first step in development of the logical side of geometry was taken by Thales, he also put his geometry to practical uses such as finding the distance from shore to a ship at sea and finding the heights of pyramids from measurements of their shadows. Stamper suggests that Thales was primarily an astronomer and may have been impelled to pursue the study of geometry because he recognized the relations between theory and practice.[21]

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[18] Plutarch, *The Life of Marcellus*, from the translation called Dryden's, corrected from the Greek and revised by A. H. Clough, Boston, Little, Brown and Co., 1905, p. 253. Quoted by Smith, in *op. cit.*, Vol. I, p. 90.

[19] Quoted from Heath, *op. cit.*, Vol. I, p. 120.

[20] Eric T. Bell, *The Queen of the Sciences*, Baltimore, The Williams and Wilkins Co., 1931, pp. 7–9.

It is perhaps futile to speculate on what would have been the historical outcome had the Greeks generally adopted the number system of Archimedes or used their ingenuity, displayed in abstract science and Utopian schemes, to put their theories into practical use. It should at least be pointed out that theory which gets too far ahead of practice may lead to stagnation just as easily as may practice which pays too little heed to theory. For example, the benefits of the discoveries of Archimedes in mechanics were long postponed, and geometry itself was kept within narrow limits, by Greek insistence on separation of theory and practice.

THE ALGEBRA OF DIOPHANTUS

Only brief mention need be made here of algebra, since little was done in this field by the Greeks beyond what had been done by the Egyptians and the Babylonians. Diophantus, the only Greek mathematician of importance in the field of algebra, lived in the third century A.D., about six hundred years after Greek geometry reached its culmination in the work of Euclid. Cajori says of Diophantus: "If his works were not written in Greek, no one would think for a moment that they were the product of a Greek mind. There is nothing in his works that reminds us of the classic period in Greek mathematics. His were almost entirely new ideas on a new subject. In the circle of Greek mathematicians he stands alone in his specialty. Except for him we should be constrained to say that among the Greeks algebra was an unknown science."[22]

Diophantus broke from the Greek practice of making geometric solutions of algebraic problems and made use of symbolic notation. He attained some proficiency in the solution of systems of simultaneous equations by a method of "tentative assumptions," as Cajori terms it, in which he assigned preliminary values to the unknowns and from the results deduced the true roots.

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[21] A. W. Stamper, *A History of the Teaching of Elementary Geometry*, Teachers College Contributions to Education No. 23, New York, Teachers College, Columbia University, 1909, p. 11.

[22] Cajori, *op. cit.*, p. 60.

Diophantus solved determinate equations of the second degree by some unknown method which gave only one root even though both roots were positive. However, as Heath suggests, the fact that Diophantus gives only one root does not necessarily indicate that he did not know another existed.[23] It has been said that the outstanding characteristic of the work of Diophantus is the lack of any generalized method, each problem being solved as a special case without reference to others. Heath is of the opinion that the methods of Diophantus are as general as is possible, considering his notation and the nature of the subject matter.[24]

MATHEMATICS IN GRECIAN EDUCATION

The period from 600 B.C. to 300 B.C., the most fruitful of Greek mathematics, was one of great political, economic, and intellectual activity. During this period the Greek city-states, Athens in particular, became cosmopolitan. The simple education of an earlier day, aimed at preparation for citizenship and consisting largely of instruction in religion, music, and physical and military training, was no longer adapted to existing conditions. So it was that, paralleling the development of geometry but not necessarily connected with its development, there was a change in the character and emphasis of Greek education. Cubberley summarizes the change as follows:

> The rigid drill of the earlier period began to be replaced by an easier and more pleasurable type of training. Gymnastics for personal enjoyment began to replace drill for the service of the state. . . . The old authors . . . began to be replaced by more modern writers, with a distinct loss of the earlier religious and moral force. . . . Education became much more

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[23] Sir Thomas L. Heath, *Diophantus of Alexandria*, Cambridge, Cambridge University Press, 1910, pp. 55–56.

[24] *Ibid.*, p. 56.

individual, literary and theoretical. Geometry and drawing were introduced as new studies. Grammar and rhetoric began to be studied, discussion was introduced, and a certain glibness of speech began to be prized. The citizen-cadet years, from sixteen to twenty, formerly devoted to a rather rigorous physical training, were now changed to school work of an intellectual type.[25]

However, Greek education was neither publicly supported nor universal. It was for the male children of citizens—not for slaves or women—and was privately supported.

According to Heath, Greek schools were differentiated into three divisions which corresponded somewhat to elementary, secondary, and collegiate divisions of today.[26] Elementary education lasted until a boy was fourteen years of age; the secondary period included the years fourteen to eighteen. Arithmetic was taught in elementary school and geometry and astronomy in secondary school. Heath suggests that, when Plato spoke of (1) calculation and the science of numbers, (2) mensuration in one, two, and three dimensions, and (3) astronomy as the three subjects fit for freeborn citizens to learn, he had in mind a selected few students, not the rank and file of Grecian youth. Plato said that there should be as much arithmetic taught "as everyone needs to learn for purposes of war, house management and civic administration; together with what it is useful for these same purposes to learn about the courses of the heavenly bodies—stars and sun and moon—insofar as every state is obliged to take them into account."[27]

Probably the Greeks added mathematics to the curriculum partly because they believed that it would aid them in learning to think clearly and partly because they considered it worth studying for its own sake. Certainly Plato believed that geometry afforded valuable training in reasoning. As Stamper suggests, Plato's Dialogues contain evidence that the Greeks were more interested in the reasoning used in geometry than in the subject matter.[28] Plato's remark that the highest value was realized from arithmetic when it was pursued in a spirit untainted by the commercialism of the Phoenicians and the Egyptians is evidence that the values ascribed to arithmetic as a school subject were other than utilitarian.[29]

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[25] E. P. Cubberley, *The History of Education,* Boston, Houghton Mifflin Co., 1920, p. 41.

[26] Heath, *A History of Greek Mathematics,* pp. 18–25.

[27] Plato, *Laws,* Book VII, 809, c-d, transl. by R. G. Bury, New York, G. P. Putnam's Sons, 1926, p. 73.

The Greeks did not limit their preference for the theoretical to mathematics alone. Philosophers like Plato pondered long over ideal forms of government, but the problem of unifying the Greek city-states into a nation remained unsolved until Alexander the Great, and after Alexander the Roman legions, transferred the problem from an ideal to a material plane.

ROMAN MATHEMATICS AND EDUCATION

The Roman people were a materialistic, matter-of-fact nation of farmers, businessmen, soldiers, and executives. They brought to practical reality the vision of world empire conceived by Alexander the Great. They centered their attention on the material world and ignored the ideal. In mathematics, the Romans did little original work. They were interested only in the utilitarian aspects of the subject.

The story of the death of Archimedes is typical of the Roman influence on mathematics. When Rome besieged Syracuse, Archimedes was stimulated to apply his knowledge of mechanics, optics, and mathematics to the practical use of constructing war engines to repel the invaders. His designs were so successful that the Romans withdrew. However, they returned later and took the city by surprise while the Syracusans were celebrating a feast to Diana. As for Archimedes, he was murdered by a Roman soldier who saw nothing of value in the circles which Archimedes insisted on preserving.[30]

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[28] Stamper, *op. cit.*, p. 17. See, for example, Plato, *The Republic*, Book VII, 526–527, transl. by B. Jowett, New York, Charles Scribner's Sons, 1902, Vol. II, pp. 354–355.

[29] Plato, *Laws*, Book V, 747, a-c, transl. by R. G. Bury, New York, G. P. Putnam's Sons, 1926, pp. 387–389. Quoted by A. E. Taylor, *Plato, The Man and His Work*, New York, Dial Press, 1936, p. 479.

The Romans made extensive use of mathematics in surveying and engineering. However, the numerous monuments to their engineering skill, scattered over the Mediterranean world, are monuments to Roman organizational and constructive ability more than to their mastery of the refinements of engineering theory. Roman roads, aqueducts, and sewers were marvelous solutions to practical problems; and the great stadiums and temples were as characteristic of the Roman people as refinements of design in the details of the Parthenon were typical of the Greeks. The outstanding contribution to theoretical mathematics during the Roman era was Heron's formula for finding the area of a triangle when the lengths of the sides are known. But Heron was a Greek living in Egypt, not a Roman. So we may say that Roman emphasis on the practical bore little fruit in the way of new mathematical theory. Certain it is that the Romans were unable to devise a number system that was usable for purposes of computation, even though there was great need for such a system.

Roman number notation served very well to enumerate things. For example, it still finds limited use in numbering chapters in books, on clock dials, and on the cornerstones of buildings. But it is totally unadapted to computational uses, as an attempt to perform the simplest multiplication using the Roman notation will demonstrate. Paradoxical as it may seem, number systems become more useful and practical as they become more abstract. The practical Romans never divorced their number system from the concrete. Hence it has no symbol for zero and makes no use of the concept of decimal place-value.

Early Roman education, like early Grecian, was largely a matter of learning through actual participation in the activities of home and community. Cubberley says, "Its purpose was to produce good fathers, citizens, and soldiers. . . . It produced a nation of citizens who willingly subordinated themselves to the interests of the state, a nation of warriors who brought all of Italy under their rule, a calculating, practical people who believed themselves destined to become the conquerors and rulers of the world, and a reserved and proud race, trained to govern and to do business, but not possessed of lofty ideals or large enthusiasms in life."[31]

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[30] Plutarch, *op. cit.*, pp. 252–259. Contains an interesting account of the war machines of Archimedes and of the circumstances leading up to his death.

With wealth and slaves obtained by conquest to relieve the Roman upper class from labor and with the poor living in idleness on a dole, a change in Roman education took place. Rome became Hellenized. Education of the youth of the land was largely given over to Greek slaves and the subject matter was largely of Greek origin. With possible exception of the period from 425 to 475 A.D., Roman education was not publicly supported and probably reached only a small portion of the people.

Arithmetic—the art of calculation—occupied a place of large importance in Roman primary schools. Cubberley mentions the emphasis placed on ability to reckon by use of fingers or counting board.[32] Roman numbers, as we have seen, did not lend themselves to the type of calculation to which we are accustomed.

Grammar and literature were the basic studies of Roman secondary schools. Above the secondary schools were the schools of rhetoric, attended by the sons of wealthy men after finishing the secondary or grammar schools at the age of sixteen. The curriculum was composed of subjects which later were to be known as the "seven liberal arts": grammar, rhetoric, dialectic, music, arithmetic, geometry, and astronomy. In sharp contrast to Greek practice, the Romans stressed practical applications of each subject. Thus, Cubberley quotes Quintilian to the effect that music was studied "largely to help with gestures and train the voice, geometry to aid in settling law suits relating to land, dialectic (logic) to aid in detecting fallacies, and astronomy to understand the movements of the heavenly bodies and the references of literary writers."[33]

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[31] Cubberley, *op. cit.*, p. 60.

[32] *Ibid.*, p. 65.

This system was very successful in training men for public service and as orators. It touched only a few, however, and failed perhaps to be broadly educative in that it did not inculcate ideals of conduct. Cubberley says that "Men educated in these schools came to boast that they could speak with equal facility on either side of any question, and the art [of oratory] came to depend on the use of many and big words and on the manners of the stage. Such ideals naturally destroyed the value of these schools and stopped intellectual progress so far as they contributed to it."[34]

Roman schools did not prepare for business, engineering, or many other great lines of activity in which the Romans excelled. They did provide, however, effective preparation for careers as public officials. At best, Roman rhetorical schools reached only a small part of the population and, despite their emphasis on the practical, stimulated neither intellectual nor material progress. Nor did they insure political stability of the Roman state.

The most striking characteristic of Roman education, aside from the fact that it was copied directly from the Greek, was the spirit of intense practicality which pervaded it. This is all the more remarkable when we consider that the Romans made use of essentially the same subject matter as did the Greeks, who stressed the theoretical and ideal. It would be the height of folly to ascribe the virtues or the vices of a people to their conception of the nature and educative value of science and mathematics, yet the fact remains that Roman education, with its overemphasis on the practical, proved to be as miseducative as the Greek with its overemphasis on the ideal. But it is significant to note that the Greeks studied mathematics for its own sake and made tremendous contributions to the knowledge of the race; the Romans, like Egyptians and Babylonians, studied mathematics for its practical uses and, when the limits of available theory were reached, progress stopped.

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[33] *Ibid.*, p. 70.

[34] *Ibid.*, p. 71.

MATHEMATICS AND EDUCATION IN THE MIDDLE AGES

With the collapse of the Roman Empire before the barbaric flood which poured in from beyond the Rhine and the Danube, education and intellectual pursuits all but vanished from Europe. The only places where any attempt was made to preserve even fragments of the learning of Greece and Rome were the monasteries, and their interest was chiefly in religious writings. In fact, the Roman church finally forbade its clergy to read pagan books, so that Greek learning, and even knowledge of the Greek language, died out in western Europe.[35]

However, the monasteries, needing priests able to read the church services, had to teach their novices to read. If neophyte priests were to be instructed in church doctrines, it was desirable that they be able to read the sacred books. Likewise, only in the monasteries were there opportunities to copy manuscripts. So the monasteries kept alive the art of reading, writing, and the mathematics which was necessary to satisfy their simple needs. So far as the need for mathematics was concerned, this amounted to being able to compute the date of Easter and little more. Later, the need for better-informed men in higher ecclesiastical posts led to development of higher schools wherein were taught the "seven liberal arts" of the Roman rhetoric schools. More advanced schools taught ethics, and all pointed toward the study of theology. Many of the schools taught only grammar, rhetoric, and logic, with chief emphasis on grammar. Only a few ventured to teach even the little mathematics known to the Dark Ages.

The study of arithmetic was still hampered by Roman notation. Arabic numerals were not brought into common use until about the time of the discovery of America, although they were known in Europe as early as 1200 A.D. According to Sanford, Roman numerals were still used in government accounts as late as Queen Elizabeth's time in England and as late as 1734 in France.[36] The study of computational arithmetic was largely a process of learning the art of finger reckoning, the use of an abacus, and other related aids to computation. The theoretical subject matter was largely what would now be termed numerology.

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[35] *Ibid.*, p. 95.

Stamper summarizes the teaching of geometry in the Middle Ages as follows: "The teaching of geometry in the church schools of the Middle Ages was limited to the learning of definitions and the performing of a few simple constructions with rule and compass. The subject was generally taught but there was no real instruction as we understand it today. The influence of Euclid was practically nil. After Gerbert there was a little work also in the finding of heights and distances, the astrolabe and mirror being the field instruments employed. The method of recitation was such that an ability to memorize was the main desideratum."[37]

Study of algebra did not go beyond the point where Diophantus left it. In algebra as in arithmetic, further progress was dependent upon development of a usable notation.

MATHEMATICS AND EDUCATION DURING THE REVIVAL OF LEARNING

Like the twilight period of the Roman Empire which preceded the intellectual night of the Middle Ages, the dawn was a period of turmoil, social unrest, and war. A whole series of events—the invention of printing, introduction of Arabic notation, discovery of America, and the religious wars of the Reformation—radically altered the nature and purpose of education both in special subjects such as mathematics and in general. The invention of printing, about 1450, made books available to increasing numbers of people. Learning ceased to be the exclusive property of religious orders.

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[36] Sanford, *op. cit.*, p. 96. See also the Introduction to C. F. Hill, *The Development of the Arabic Numerals in Europe*, London, The Clarendon Press, 1913.

[37] Stamper, *op. cit.*, p. 47.

Rediscovery of the works of ancient Greek writers, such as the *Elements* of Euclid, added to the meager store of learning available to scholars. Along with knowledge of Euclid's work, the Moors brought into Europe by way of Spain the Hindu-Arabic numerals which freed computation from such mechanical aids as the abacus. Expanding sea travel—brought about in part by Moslem conquest, in part by increasing trade with other lands, and in part by improved instruments for navigation such as the magnetic compass—opened up new uses for mathematics. Finally, the religious wars of the Reformation, with the new philosophy concerning the relation between the individual and the church, led to a new need for education.

During the Reformation a major part of the present-day curriculum in secondary-school algebra was developed. Building on the work of Diophantus and using the new Hindu-Arabic numerals, men such as Cardan, Tartaglia, Ferrari, Vieta, Pascal, Descartes, and Fermat, to name only a few, developed the ideas on which the solutions of equations, notations, and processes of present-day algebra are based. It will be sufficient here to sketch only the main contributions of some of these men, remembering that the work of Diophantus represented essentially the total achievements of the Greeks in algebra and that he could solve linear, quadratic, and special cases of cubic equations.

Vieta pioneered in the use of literal numbers. Cardan and Tartaglia (1545) are remembered in connection with solution of the general cubic. Ferrari (1545) first solved the general biquadratic. Pascal, among other things, developed part of the theory of the binominal theorem (1654) and made contributions to geometry and the theory of probability. Descartes and Fermat (1637) developed analytical geometry, uniting algebra and geometry. Newton and Leibnitz independently developed the theory of differential and integral calculus in the years after 1660. Napier invented logarithms (1614).

The history of development of the theory of probability is of interest in showing how a mathematical science may originate in the most mundane things, then attract the interest of gifted men who study it for its own sake, and finally be put to uses of which neither they nor those who first proposed the problem ever dreamed. Considerable work had been done in permutations and combinations, when the gamblers of the Renaissance interested mathematicians in the problem of calculating their chances of winning. Cardan was an inveterate gambler himself and wrote a book on games of chance. Pascal and Fermat, in 1654, became interested in the "problem of points." This problem is essentially as follows: "Two players want each a given number of points in order to win; if they separate without playing out the game, how should the stakes be divided between them?"[38] Huygens, Bernoulli, De Moivre, and Laplace, among others, contributed to the theory of probability, De Moivre's studies leading to discovery of the normal probability curve.[39] Halley, the astronomer, applied the theory of probability to construction of a mortality table and thereby opened the way for modern theories of life insurance. Thus the theory of probability arose in the solution of problems from the fields of gambling and astrology—fields, by modern standards at least, of morally questionable respectability. It became respectable on being studied for its own sake, and finally went on to the useful labor of taking the gamble out of certain processes in industry and out of the problem of providing for security in old age.

In arithmetic, the great development of the Renaissance was away from such mechanical aids to computation as the abacus and finger counting, towards methods familiar to us today. This development was made possible by the introduction of Hindu-Arabic numerals utilizing the concepts of place-value and zero. The theory of decimal fractions was brought into usable form, except for a good notation, in 1585 by Simon Stevin. The arithmetic of this period was largely commercial, since developing trade, and particularly the lifting of the ban against interest, created a demand for solution of the typical problems of commerce.

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[38] I. Todhunter, *A History of the Mathematical Theory of Probability*, London, Macmillan and Co., 1865, p. 8. See also D. E. Smith, *A Source Book in Mathematics*, New York, McGraw-Hill Book Co., 1929, pp. 545–565.

[39] See Smith, *A Source Book in Mathematics*, pp. 566–575, for a reprint of De Moivre's paper, "On the Law of Normal Probability," edited by Helen M. Walker.

But of even greater importance was the effect of the Protestant Reformation on educational ideas of the time. It must be remembered that no schools prior to the Reformation affected more than a small minority of the people. In Egypt and Babylonia only neophyte priests, and possibly the sons of wealthy nobles, went to school. In Greece only the sons of citizens, again a minority, were educated. In Rome only the sons of the wealthy had educational opportunities, and the church schools of the Middle Ages were primarily for those who expected to enter holy orders. The church of the Middle Ages was an authoritarian institution. The individual was told what to believe and was at times vigorously dissuaded from attempting to think for himself, as in the case of Galileo whose treatment was mild compared to that received by certain luckless "protestants." Obviously, in such an organization, it was necessary to educate the priesthood alone; the common people needed only to listen to what they were told.

The Protestants faced another situation. The individual was charged with responsibility to decide for himself, hence it was essential that each individual be taught to read his Bible if he were to decide for himself questions of faith and conduct. Here for the first time we find the idea of the common school—education for all the people—a concept that has grown into the complex public educational system of today. The first common schools were church supported and controlled. Their aim was definitely utilitarian: to teach youth to read that they might learn their duty to the church. This was education imposed from without, education centered on adult rather than child needs. Although it was undisguised indoctrination, it was still the beginning of education for all. Likewise, of great importance was the fact that instruction in these schools was in the vernacular, not in Latin as in earlier church schools. The Catholic church held itself to be superior to the secular state but the Protestant churches were, in effect, part of the state.

SUMMARY

Discussion up to this point may be summarized as follows: The history of Occidental mathematics to the time of the Reformation may well be divided into four periods. In the first the dominant culture was that of Egypt and Babylonia. Only the utilitarian values of mathematics were recognized and education was confined largely to preparation for the priesthood.

Greece dominated the second period. The Greeks emphasized the theoretical aspects of mathematics and neglected the practical. In marked contrast to the stagnation which resulted from Egyptian and Babylonian emphasis on the practical, the Grecian era of emphasis on the theoretical was characterized by great progress in mathematics, particularly in the field of geometry. Education in Greece was limited to the sons of citizens and was cultural rather than practical.

In the third period emphasis was on the practical both in mathematics and in education. During the Roman era great progress in engineering practices resulted from practical applications of Greek mathematics, but no theoretical advance of note was achieved. The practical Romans, like the idealistic Greeks, failed to devise a usable number system, and Roman education proved to be better adapted to training unprincipled political opportunists than to developing statesmen.

In so far as progress in mathematics was concerned, the Middle Ages was a continuation of the third period. Emphasis was still on the practical, to the exclusion of the theoretical, and the monastic schools failed to preserve even as much of the spirit of Greek learning as did the Romans. Again, as in early Egypt, education became a matter of training for the priesthood.

The Renaissance ushered in the fourth period, one during which learning was again sought for its own sake. Mathematical developments of immense value were the immediate fruits of the Revival of Learning. Hindu-Arabic notation—the ideas of zero and of place-value of numbers—revolutionized the art of computation. Modern arithmetic and algebra attained completeness of development comparable to that of geometry during the Grecian period. The Reformation created a need for universal education.

One seems forced to conclude that greatest intellectual progress is achieved by consideration of the theoretical. However, since mathematics grew out of human needs, neglect of the practical may lead to theoretical results of little or no lasting importance or value. If stagnation is to be avoided, proper blending of the theoretical and the practical appears to be essential.

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Chapter 3

American Background of Today's Curriculum

THE bitter wars which broke out in Europe after the Reformation, with their accompanying religious intolerance, led to settlement in America of oppressed minorities fleeing from persecution in their homelands. This meant that the early settlers in what is now eastern United States were predominantly Protestant, hence were a people who felt the need for schools. As Cubberley says, "Education of the young for membership in the church, and for the perpetuation of a learned ministry for the congregations, from the first elicited the serious attention of these pioneer settlers."[1]

Further, despite the large numbers of members of other denominations who came to America, such as Quakers to Pennsylvania, Catholics to Maryland, and Anglicans to Virginia, it was the Calvinistic Puritans of Massachusetts and New England who really gave direction to the early development of education in the United States. It was in Massachusetts that the first elementary schools in the United States were started, the first secondary schools, and the first college. In Massachusetts, too, we find for the first time education a concern of the state, even though at first the state was a creature of the church.

[1] E. P. Cubberley, *The History of Education*, Boston, Houghton Mifflin Co., 1920, p. 357.

THE FIRST PUBLIC SCHOOLS IN AMERICA

The system of education as first set up in Massachusetts was voluntary, with elementary instruction in the homes, a grammar school in the towns to prepare boys for college, and, after 1636, Harvard College to prepare them for the ministry. But many parents neglected to educate their children properly, so we find the Massachusetts Law of 1642 charging town officials with the duty of seeing that children were trained "in learning and labor and other employment . . . profitable to the Commonwealth."[2] They were also to be taught to read and to understand the principles of religion and the laws of the country. The selectmen were empowered to fine parents who did not provide for proper education of their children. In 1645 a law was passed providing that youths between ten and sixteen years of age be given instruction in the use of arms "as small guns, halfe pikes, bowes, and arrowes, etc." In 1647 a law, often termed from the wording of the preamble the "Old Deluder Satan Law," made it compulsory for towns of fifty families or more to establish an elementary school and for larger towns to have a secondary school. These laws contain the germ of the American public-school system.[3] It is important to note that the schools were not set up in the interests of children, but to serve church and state.

The curriculums of the schools varied from town to town but were largely devoted to reading and religious instruction. Monroe is of the opinion that religion kept the elementary subjects as we now know them, particularly arithmetic, out of the schools.[4] However, he also cites instances in which arithmetic was taught, so it evidently was not omitted entirely. Monroe points out that in New York the commercial interests of the Dutch made the teaching of arithmetic imperative. In all colonial schools arithmetic, if taught, was taught primarily as a tool of commerce. In some cases, particularly in Boston, a double system of schools was established; one school taught reading, the other writing and arithmetic. Sometimes the writing-and-arithmetic schools were private, and supplemented the instruction offered in the public grammar school.

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[2] *Ibid.*, p. 364.

[3] *Ibid.*, p. 366.

[4] Walter S. Monroe, *Development of Arithmetic as a School Subject*, U.S. Bureau of Education Bulletin No. 10, 1917.

The typical school of this period might be said to have been a grammar school and was, in effect, a secondary school. Elementary schools were not established until the lack of preparation of students wishing to enter the established grammar schools made it evident that formal elementary schools were necessary. Previous to their establishment elementary education—teaching children to read, write, and possibly to count—was a function of the home. Schools took over this function only when the home failed to perform it satisfactorily, much as the schools of today are taking over still other functions formerly performed by home and church.

These secondary schools, variously called grammar schools, Latin schools, Latin grammar schools, or even free schools, were largely imitators of English schools of the same names. The English schools, in turn, were direct descendants of the monastic schools and had much the same objectives. The first secondary school in America, the Boston Latin School, was organized in 1635. Brown gives as its distinguishing characteristics that it was a town school and that its purpose was preparation for college.[5] Broome says "The aim of the grammar school of the seventeenth century was to prepare for college; the aim of the college was to supply the people with an enlightened clergy; the course of study was doubtless regarded a most efficient one for the purpose; a thorough grounding in the classics was an adequate preparation for the college course as it stood; in short, there was excellent adjustment all along the line."[6]

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[5] E. E. Brown, *The Making of Our Middle Schools,* New York, Longmans, Green and Co., 1921, p. 37.

[6] E. C. Broome, *A Historical and Critical Discussion of College Admission Requirements,* Columbia University Contributions to Philosophy, Psychology and Education, Vol. XI, Nos. 3 and 4, New York, The Macmillan Company, 1902, p. 24.

MATHEMATICS IN COLONIAL SCHOOLS

As Broome points out, admission to college in colonial days required rather thorough knowledge of the classics and little else.[7] Therefore, mathematics was largely neglected in colonial grammar schools. However, as mentioned above, some arithmetic was taught and we find that "The most important addition to college admission requirements during the colonial period was mathematics to the extent of elementary arithmetic. This subject appeared for the first time among the subjects for entrance to Yale College established by the revised code of 1745. . . . For the remainder of the century no mathematics beyond vulgar arithmetic was required for admission to any college; nor did Harvard include arithmetic until after 1800."[8]

In colonial days, instead of being required for admission to college, arithmetic, algebra, and geometry were college subjects. Thus, about the middle of the seventeenth century, the schedule of studies at Harvard listed a forty-five-minute period once a week of arithmetic and geometry for "senior sophisters." After 1655, when the course was extended to four years, mathematics was taught in the fourth year. By 1744, Yale had moved geometry down to the sophomore year; Harvard followed in 1787. By 1818 geometry was a first-year subject.

As early as 1753, the college division of the institution which later became the University of Pennsylvania taught algebra, geometry through the first six books of Euclid, and arithmetic, all of them the first year. The second year was devoted to algebra, "the remainder of the first six books of Euclid, together with the eleventh and twelfth, and also the elements of geometry, astronomy, chronology, navigation, and other most useful branches of mathematics."[9]

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[7] *Ibid.*, p. 32.

[8] *Ibid.*, p. 30.

[9] Quoted from *The Works of William Smith*, D.D., Late Provost of the College and Academy of Philadelphia, Vol. I, Part II, p. 183, by Lao G. Simons, *Introduction of Algebra into American Schools in the Eighteenth Century*, U.S. Bureau of Education Bulletin No. 18, 1924, p. 50.

The arithmetic taught in colleges was essentially the science of numbers and, except at Franklin's academy, practical applications were ignored. This was not the situation in the lower schools of colonial days. Simons says:

> In the American Colonies arithmetic was an important subject for practical reasons. It was needed for trade and commerce. With sailing vessels plying between Europe and America and the only means of communication with the "homeland," navigation and all the kinds of sailing that had to be put to daily use came to be a continuation of the course in arithmetic. Astronomical observations were an important feature in laying out a course at sea, and so astronomy is found in connection with arithmetic. Some elementary trigonometry, logarithms and geometric constructions played a necessary part in the calculations incident to both navigation and astronomy. With this list the practical uses of mathematics in that day are exhausted.[10]

On the other hand, as Broome points out, "The education of the colonial college was for a privileged class who desired humanistic culture, not for the common toilers who needed useful knowledge for the practical affairs of life."[11] Hence college instruction in mathematics emphasized the theoretical and philosophical aspects of the subject. Simons says of algebra:

> Nowhere are there found indications that a practical need for algebra actuated the teaching of it during this [the colonial] period. The inclusion of this subject in the curriculum of a college of the eighteenth century, or the teaching of it as a special subject by some enthusiastic teacher must be accounted for on the ground that it was done for the sake of the subject itself or for the theoretical aspect of fluxions. The fascination of this kind of analysis attracted teacher and pupil alike, and the simple joy of the intellectual life that it afforded was reward enough for its study, a reason that lies at the very heart of progress along any line of mental activity.[12]

TEACHING METHODS

The methods of instruction used in teaching mathematics in colonial colleges are also evidence that practical applications of the subject were not intended. In the teaching of neither arithmetic nor algebra was drill employed to insure mastery of fundamental skills. The teaching of geometry, as in seventeenth-century England, was of the extreme Euclidean type.

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[10] Simons, *op. cit.*, p. v.
[11] Broome, *op, cit.*, p. 34.
[12] Simons, *op. cit.*, p. 75.

According to Simons, algebra notebooks written by students at Harvard and Yale in the years just preceding the Revolutionary War indicate that the instructor merely dictated a treatise on algebra to the class. Apparently there was no drill on techniques and little or no problem solving.

Arithmetic was similarly taught. Monroe says, "The immediate end sought, which also represents the standard of instruction, was a knowledge of the rules and their application. . . . The pupil was expected to learn the rule and then apply it to a very few examples or problems. No opportunity was given for drill upon the applications of the rule, even in the case of the fundamental operations. Skill and facility were not expected nor attempted."[13]

The method of instruction used was that of the ciphering book. A pupil ambitious to learn arithmetic provided himself with a blank-book, much like the account books or student notebooks of today, and took it to the teacher—if he were fortunate enough to have a teacher who was capable of giving him instruction in "ciphering." The teacher assigned a problem, taken from his own ciphering book or from one of the arithmetic texts in use at that time, such as Dilworth's *Schoolmaster's Assistant,* Pike's *Arithmetic,* or Daboll's *Schoolmaster's Assistant.* Having set the problem, the teacher gave the rule and the novice was then expected to work out his own solution. The following description of the process is from Barnard's *American Journal of Education:*

> At length, in 1790 or 1791, it was thought that I was old enough to learn to *cypher* and accordingly was permitted to go to school more constantly. I told the master I wanted to learn to cypher. He set me a sum in simple addition—five columns of figures and six figures in each column. All the instruction he gave me was, add the figures in the first

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[13] Monroe, *op. cit.,* p. 16.

> column, carry one for every ten, and set the overplus down under the column. I supposed he meant by the *first* column the left hand column, but what he meant by carrying one for every ten was as much a mystery as Samson's riddle was to the Philistines. I worried my brains for an hour or two and showed the master the figures I had made. You may judge what the amount was when the figures were added from left to right. The master frowned and repeated his former instruction—added [sic] up the column *on the right*, carry one for every ten, and set down the remainder. Two or three afternoons (I did not go to school in the morning) were spent in this way, when I begged to be excused from learning to cypher, and the old gentleman with whom I lived thought it was time wasted. . . . The next winter there was a teacher more communicative and better fitted for his place, and under him some progress was made in arithmetic, and I made a tolerable acquisition of the first four rules, according to Dilworth's *Schoolmaster's Assistant,* of which the teacher and one of the eldest boys had each a copy. The two following winters, 1794 and 1795 . . . I mastered all the rules and examples in the first part of Dilworth; that is through the various chapters of Rule of Three, Practice, Fellowship, Interest, etc., etc., to Geometrical Progression and Permutation. [Italics in the original.][14]

As the foregoing indicates, texts were used when available. This was a means of saving time, since it was then no longer necessary for the master to dictate problems and rules. The quotation also hints at the mediocre qualifications possessed by some teachers of that period. Cajori tells the story of a young teacher who, as part of an examination to determine his fitness to teach, was asked, "What is the product of twenty-five cents times twenty-five cents?" Both applicant and examiner agreed that the answer was six and one-fourth cents. Although such a small sum did not look right, they were unable to find a rule in Pike's *Arithmetic* which indicated that there was anything wrong with their solution. It is obvious that neither applicant nor examiner had a very thorough understanding of the fundamental operations with denominate numbers.

Since the primary purpose of colonial colleges was to train clergymen, major emphasis was on the study of languages. Mathematics occupied a minor place in the college curriculum and only the theoretical aspect was considered. In the grammar schools mathematics was not part of the regular curriculum but was taught in an auxiliary school. In the lower schools the utilitarian values of mathematics were stressed. However, the methods of teaching mathematics in colonial schools were such that neither aim was effectively realized.

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[14] Barnard's *American Journal of Education,* 13:130 (March, 1863). Quoted by Monroe, *op. cit.,* p. 44.

RISE OF THE ACADEMIES

In the last half of the eighteenth century there was, as Broome suggests, a momentous change in the religious, social, and political conditions in the country. He continues, "With the development of political independence came democracy and a pretty general fusing of classes. Naturally new educational demands arose. There was a call for higher education in subjects of practical value—'bread and butter subjects.' . . . The colleges, however, were not immediately responsive to the new demands. . . . The demand for popular and useful studies was met, however, by the rise of a new class of institutions—the academies."[15]

The academies offered instruction in subjects such as geography, astronomy, algebra, geometry, trigonometry, surveying, navigation, and history. The first academy was founded at Philadelphia in 1750 largely as a result of the efforts of Benjamin Franklin, who proposed that the students "learn those Things that are likely to be most useful and most ornamental."[16] Reference has already been made to the collegiate division of this institution. The academy, as an educational institution, soon displaced the Latin grammar schools and became the typical secondary school in America during the formative period of the nation.

Brown considers the academy as marking the transition from the stratified society of colonial times to modern democracy.[17] The academy sought to provide education beyond the elementary school for children of middle-class parentage who did not expect to enter college. Because the academies were intended to be terminal schools, the subject matter taught and the spirit of the instruction differed from those of the Latin grammar schools. Brown says, "There was, in the academies, a growing sense of the value of education for its own sake, or rather for its effect in heightening sheer human worth."[18] In other words, the academy was concerned with the individual as a competent member of society, vocationally and otherwise, and not with preparing him to serve church and state.

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[15] Broome, *op. cit.*, pp. 40–41.

[16] Benjamin Franklin, *Proposals Relating to the Education of Youth in Pennsilvania*, facsimile reprint, Philadelphia, University of Pennsylvania Press, 1931, p. 11.

[17] Brown, *op. cit.*, pp. 228–229.

THE ACADEMY CURRICULUM

Although the academies taught subjects which were required for college entrance, they added new subjects without reference to college-entrance requirements. Brown mentions studies in English, history, and natural science as the most significant of these additions. He says further, "The first stage in the introduction of natural science into the program of studies is seen in the laying of strong emphasis on mathematics, especially algebra and geometry. Closely connected with these subjects was astronomy."[19]

Physics, chemistry, and geography were included in the subjects introduced under the heading of natural philosophy. According to Brown, these subjects were studied partly for their own sake but more because they were deemed useful. He says, "In all the studies of this group, the speculative and liberal interest ran alongside of the consideration of practical use—sometimes one ahead, and then again the other. To the general public such subjects doubtless appealed on account of some sort of usefulness. Their practical value was sometimes emphasized by the addition of technical instruction in surveying and navigation after the example of a few colonial schools."[20]

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[18] *Ibid.*, p. 229.
[19] *Ibid.*, p. 232.
[20] *Ibid.*, p. 233.

MATHEMATICS IN THE ACADEMY CURRICULUM

A good idea of the place of mathematics in the academy curriculum is given by the course of study for Phillips Exeter Academy for 1818.[21] Instruction was offered in two departments—the college-preparatory or classical, and the English department. The college-preparatory course was four years in length and the major subjects taught were Greek and Latin. Arithmetic was taught in each of the first two years, algebra in the third year, and both algebra and geometry in the fourth year.

In the English department it was specified that a candidate seeking admission must be at least twelve years of age and, among other things, must be "familiarly acquainted with arithmetic, through simple proportion with the exception of fractions." To complete the course of study in the English department required three years. Arithmetic and algebra "through simple equations" were taught the first year. The second year the pupil studied "geometry; plane trigonometry and its applications to heights and distances; mensuration of superficies and solids." In the third and final year, surveying and navigation were offered.

Since the academy had its origin in the demand for higher education with emphasis on the practical, it seems likely that the mathematics offered in the first two years in the English department was regarded as preparatory for the surveying and navigation—the bread-and-butter subjects—of the last year.

The mathematics curriculum in the academies was but an extension of the mathematics offered in the writing-and-arithmetic schools connected with the old Latin grammar schools. Brown says that when commercial and mathematical subjects came into the curriculum of the writing-and-arithmetic schools they "had no connection with preparation for college. They represented the intrusion of a different view of the function of the school. They smacked of trade. The notion that they might have some sort of educational value in and of themselves was not then abroad. Education in its several aspects was viewed as something institutional and practical. It was not for the perfecting of human character, but for the training up of men to some sort of efficiency and public usefulness."[22] It seems reasonable to assume that these remarks of Brown are equally applicable to academy curriculums since the academy itself was an outgrowth of the educational philosophy Brown mentions.

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[21] *Ibid.*, pp. 237–238.

In the period immediately following the War of 1812, when Phillips Exeter was offering instruction in surveying and navigation, the westward movement was in full swing and there was great demand for surveyors to lay out new lands in the West. About a decade later we find Abraham Lincoln carrying a surveyor's chain and compass. Parenthetically, it may be noted that Washington was doing the same type of work at the time that Franklin was laying plans for the opening of the first academy. In the years following the War of 1812 the American merchant marine was in its days of glory. The boys who were studying navigation would use it on the whalers, packets, and clipper ships that carried the American flag into every port of the seven seas. No wonder the mathematics taught in the schools which were not college preparatory "smacked of trade."

It is possible that in the college-preparatory or classical departments of the academies the theoretical phase of mathematics was stressed and that in the non-preparatory departments the practical phase was emphasized. Thus, in the classical course at Phillips Exeter no mention was made of any prerequisite mathematics. Two years were devoted to arithmetic, and neither algebra nor geometry was offered until the third and fourth years respectively. Algebra and geometry thus were not preparatory for other mathematics courses nor were they prerequisite for college entrance. On the other hand, a student at Phillips Exeter who was not preparing to enter college was required to begin the study of algebra the first year and geometry the second. Some arithmetic was required as a prerequisite to entrance, and applied mathematics occupied an important place in the curriculum. Thus, in striking contrast to present practice, Phillips Exeter required more mathematics of those students who expected to terminate their formal education at the end of their academy course than it did of those preparing to enter college. This fact may indicate a utilitarian conception of the function of mathematics in education, or it may be that Phillips Exeter required less mathematics of its college-preparatory students because algebra and geometry were taught in college.

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[22] *Ibid.*, pp. 134–135.

Another factor which may have influenced the development of the academy curriculum in mathematics needs to be mentioned. If, as was largely true of colonial colleges, only the few were privileged to attend and study the typical college subjects, then knowledge of those subjects would be coveted for the social prestige it afforded. Because the effects of this factor are not readily identified, it is easily overlooked.

Later the trend of curricular development in mathematics in the academy doubtless turned toward the theoretical because the teachers were largely college trained and tended to stress the theoretical aspects of mathematics because their college instructors had done so. As a result, there was a natural tendency to lose sight of the utilitarian aim and it is quite probable that the mathematics courses of the academy soon became quite abstract in content and spirit and also that whatever differences there may originally have been between the mathematics in the terminal courses and that in the college-preparatory courses soon disappeared.

In summary, it seems evident that mathematics first came into American secondary-school curriculums because of its practical utility and not because it was a prerequisite for entrance into college. The Revolutionary period was one of commercial expansion. There was a demand for training in surveying, navigation, and the arithmetic of commerce. Consequently algebra and geometry ceased to be college subjects and were taught in the secondary school. Arithmetic became an elementary-school subject. At the same time factors such as the social prestige of college mathematics and the fact that teachers were trained in abstract mathematics tended to make formal and abstract the mathematics taught in the academy. Apparently it was only after training in arithmetic, and later in algebra and geometry, had become so common that it was no longer worth while for the colleges to maintain classes in these subjects that elementary mathematics became a prerequisite for college entrance.

CHANGING TIMES CALL FOR CHANGES IN SCHOOLS

The rise of the academies marked the end of the religious domination of education that had endured since the overthrow of the Roman Empire. The great contributions of church schools had been the ideas of universal literacy and instruction in the vernacular. Their great weakness was that they put too great emphasis on grammar and the classical languages.

Educational institutions and curriculums become outmoded, not by time, but because social, political, and economic forces change the conditions of life. The rise of the academy was not an isolated phenomenon but the result of forces which created fundamental changes in the civilization of the time. The academy, as has been said, freed the school from religious domination. The American Revolution, which was only one phase of a world-wide movement for political reorganization, completed the separation of church and state in America. The trend of the times was toward greater emphasis on the importance of the individual. In American politics, this movement culminated in the contest between the Jacksonian Democrats and the more aristocratic Whigs which resulted in the ascendancy of the individualistic Democrats.

NEW CONCEPTS OF EDUCATION

In education, the counterpart of the struggle for greater political liberty was the struggle to liberalize education and to free it from the shackles of overwhelming domination by the past. The educational philosophy of the time was greatly influenced by the ideas of John Locke and of Jean Jacques Rousseau. It may seem strange to mention Locke and Rousseau in connection with the democratic movement in education, when it is remembered that their proposals for new programs of education applied only to the education of the sons of gentlemen. However, their ideas of education, greatly modified in many respects, are still basic in modern educational philosophies.

John Locke was, for example, the founder of modern psychology and believed that the purpose of education was, above all, to make man a reasoning creature. He believed that it was the process of learning that was important, rather than the thing learned. He is often charged with being the father of the doctrine of formal discipline. Rousseau, on the other hand, was an exponent of the laissez-faire philosophy of education—the advocate of "let nature take its course." Hence, an understanding of the issues of modern education requires an understanding of the ideas of these two men.

JOHN LOCKE ON EDUCATION

Locke believed that no subject could develop the ability to reason as well as mathematics. His views concerning the place of mathematics in education, then, are of particular significance here. He said:

> Would you have a man reason well, you must use him to it betimes, exercising his mind in observing the connexion of ideas and following them in train. Nothing does this better than mathematics, which therefore I think should be taught to all those who have the time and opportunity, not so much as to make them mathematicians as to make them reasonable creatures; for though we all call ourselves so because we are born to it if we please, yet we may truly say, nature gives us but the seeds of it; we are born to be, if we please, rational creatures, but it is use and exercise only that makes us so, and we are indeed so no farther than industry and application has carried us. . . . I have mentioned mathematics as a way to settle in the mind an habit of reasoning closely and in train; not that I think it necessary that all men should be deep mathematicians, but that, having got the way of reasoning which that study necessarily brings the mind to, they might be able to transfer it to other parts of knowledge as they shall have occasion. For in all sorts of

reasoning every single argument should be managed as a mathematical demonstration; the connexion and dependence of ideas should be followed, till the mind is brought to the source on which it bottoms, and observes the coherence all along.[23]

Obviously, Locke was concerned with the theoretical aspects of mathematics and not with the practical. But he did not stress mastery of subject matter "to make them mathematicians"; he wished to develop reasoning ability through the study of mathematical subject matter.

Locke's concept of the function of mathematics in education was a natural consequence of his ideas on psychology. He built up a theory of psychology based on the analysis of experience and maintained that all knowledge is derived from two sources: (1) external material things as perceived by the five senses and (2) internal experiences derived from mental activity. He compared the mind to a wax tablet upon which impressions are made by experiences as by a stylus.

Locke denied the existence of innate ideas—the basis of Greek educational theory—and so "paved the way for the idea of human development."[24] Eby and Arrowood summarize the educational implications of the doctrine of innate ideas as follows: "The doctrine of innate ideas implies that the mind of the child possesses the same fundamental intellectual furniture that is found in the mind of the adult. In the one, knowledge is still implicit though as yet unrealized; in the other it is fully explicit and understood. To all intents and purposes the child is a miniature adult and simply expands into adulthood. If ideas are not innate, then the child differs radically from the adult, so far as his intellectual possessions are concerned."[25]

Locke's denial of the doctrine of innate ideas therefore destroyed the philosophical basis of the old classical education. It also gave voice to a belief in the possibility of realizing unlimited potentialities in every individual through proper education. In short, as Eby and Arrowood have indicated, it made room for the idea of human development and betterment through education. Denial of the doctrine of innate ideas also led directly to the concept of education based on the needs and interests of the child as opposed to adult needs and interests and, therefore, directly to the educational theories of Rousseau.

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[23] John Locke, "Of the Conduct of the Understanding," *The Educational Writings of John Locke*, John W. Adamson, ed., New York, Longmans, Green and Co., 1912, pp. 196–199.

[24] Frederick Eby and C. F. Arrowood, *The Development of Modern Education*, New York, Prentice-Hall, Inc., 1934, p. 394.

[25] *Ibid.*, p. 395.

THE EDUCATIONAL THEORIES OF ROUSSEAU

The keynote to Rousseau's theory of education is found in the opening sentence of *Emile:*

> Everything is good as it comes from the hands of the Author of Nature; but everything degenerates in the hands of man. He forces one country to nourish the productions of another; one tree to bear the fruit of another. He mingles and confounds the climates, the elements, the seasons; he mutilates his dog, his horse, his slave; he overturns everything; disfigures everything; he loves deformity, monsters, he will have nothing as nature made it, not even man; like a saddlehorse, man must be trained for man's service—he must be made over according to his fancy, like a tree in his garden.[26]

Rousseau emphasized natural development of the child. He maintained, in opposition to the theological doctrine of human depravity, that nature is good and that the way to train a child is to follow the dictates of nature. He popularized Locke's theory that the function of education is to develop the natural powers and abilities of an individual; to develop a reasoning being rather than to transmit a body of information or to teach conformity to an established pattern of behavior. Rousseau, like Locke, stressed the idea of using a child's natural activities and of educating through the senses rather than through the memory.

[26] Jean Jacques Rousseau, *Emile,* transl. by Wm. H. Payne, New York, D. Appleton and Co., 1901, p. 1.

PESTALOZZI

The work of Pestalozzi was, in reality, an attempt to put some of Rousseau's theories into practice. His labors led to modern, secular, vernacular elementary schools. Pestalozzi looked upon education as a process of growth. He likened the development of a child to the unfolding of a plant or flower. He wrote, "All instruction of man is then only the art of helping Nature to develop in her own way; and this art rests essentially on the relation and harmony between the impressions received by the child and the exact degree of his developed powers."[27] In another place Pestalozzi says, "I find I have fixed the highest, supreme principle of instruction in the recognition of *sense impression as the absolute foundation of all knowledge.*" (Italics in the original.)[28]

According to Pestalozzi, there are three elementary means of gaining knowledge: form, sound, and number. The latter, of course, means arithmetic. Of the three, arithmetic is particularly important. According to Pestalozzi:

> Now while sound and form lead us, by several subordinate methods, to the clear ideas and mental independence which we aim at through them, arithmetic is the only means of instruction which is connected with no subordinate means. Whenever it applies it appears only as a simple result of that elementary faculty by which we make ourselves clearly conscious of the relations of more or less in all seen objects and are able to represent these ratios with infinite accuracy.
>
> Sound and form very often carry seeds of error and deception in themselves—number never. It alone leads to certain results, and if measurement makes the same claim, it can only support it by the help of arithmetic and in union with it. That is, it is sure because it calculates.
>
> Now arithmetic is to be considered the means that aims most directly at the end of instruction, clear ideas, the most important of all means. It is obvious therefore, that this subject should always be pursued with special care and skill.[29]

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[27] J. H. Pestalozzi, *How Gertrude Teaches Her Children*, transl. by Lucy E. Holland and Frances E. Turner, Syracuse, New York, C. W. Bardeen 1894, pp. 25–26.

[28] *Ibid.*, p. 139.

[29] *Ibid.*, p. 132.

Of the nature of arithmetic, Pestalozzi says:

> Arithmetic arises entirely from simply putting together and separating several units. Its basis, as I have said, is essentially this. *One and one are two, and one from two leaves one.* Any number, whatever it may be, is only an abbreviation of this natural, original method of counting. But it is important that this consciousness of the origin of relations of numbers should not be weakened in the human mind by the shortening expedients of arithmetic. It should be deeply impressed with great care on all the ways in which this art is taught, and all the future steps should be built upon the consciousness, deeply retained in the human mind, of the real relations of things which lie at the bottom of all calculation. If this is not done, this first means of gaining clear ideas will be degraded to a plaything of our memory and imagination and will be useless for its essential purpose. [Italics in the original.][30]

The above quotations outline Pestalozzi's educational theories and his ideas concerning the nature of arithmetic and its place in elementary education. The relationship between his idea of sense impression as the basis of all knowledge and the ideas of Locke and Rousseau is readily evident. I have quoted at length from Pestalozzi's work because if we are to have an intelligent conception of how the present curriculum in mathematics came into being, we must understand just what was the nature of Pestalozzi's contribution. It cannot be too strongly emphasized that basic points in the educational philosophy of Pestalozzi and his followers were the conceptions of education as growth—not preparation for adult life—and of education as a process of developing the capacities of an individual more or less for their own sake, instead of preparing him for service to church and state. The idea of sense impression as the basis of all knowledge shifted the emphasis in education from memorization of factual material to the development of an experimental basis for clear understanding of principles.

Obviously, these conceptions were widely at variance with the educational practices of the ciphering-book era in mathematics education. Monroe quotes a passage from a book on Pestalozzi published in 1831 which makes this difference quite evident:

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[30] *Ibid.*, p. 133.

In this calculating world shall we be understood if we say that Pestalozzi's arithmetic had no reference to the shop or counting house; that it dealt not in monies, weights, or measures; that its interests consisted entirely in the mental exercise which it involved and its benefit in the increase of strength and acuteness which the mind derived from that exercise?

Again, in this mechanical sign-loving age, shall we be understood if we say that his arithmetic was not the art of handling and pronouncing cyphers, but the power of comprehending and comparing numbers? And, lastly, with a public whose faith is exclusively devoted to what is immediately and palpably "practical and useful," shall we be believed if we add that, notwithstanding the apparently unpractical tendency of Pestalozzi's arithmetical instruction, he numbered among his pupils the most acute and rapid "practical arithmeticians"?

Such, however, was the case; his course of arithmetic excluded cyphers until numbers were perfectly understood, and the rules of reduction, exchange, and others, in which arithmetic is applied to the common business of life, were superadded at the close of his arithmetical course as the pupil's future calling might require it. The main object of his instruction in this branch of knowledge was the development of the mental powers; and this he accomplished with so much success that the ability which pupils displayed, especially in mental arithmetic, was the chief means of attracting the public notice to his experiments.[31]

THE WORK OF WARREN COLBURN

Pestalozzian principles were applied to teaching arithmetic in the schools of New England by Warren Colburn, who in 1821 published a book entitled *First Lessons in Arithmetic on the Plan of Pestalozzi.* Colburn had not studied under Pestalozzi himself but had been influenced by descriptions of the latter's work. Some of the material in Colburn's book was borrowed directly from Pestalozzi, other parts were the fruits of Colburn's own teaching experiences.

Colburn considered the utilitarian phase of arithmetic to be highly valuable from an educational point of view but also stressed arithmetic as a "discipline of the mind." The following excerpt from an address by Colburn expresses his ideas clearly: "Arithmetic, when

[31] Monroe, *op. cit.*, p. 60.

properly taught, is acknowledged by all to be very important as a discipline of the mind; so much so that even if it had no practical application which should render it valuable on its own account, it would still be well worth while to bestow a considerable portion of time on it for this purpose alone. This is a very important consideration, though a secondary one compared with its practical utility."[32]

Colburn believed that a child's study of arithmetic should begin at five or six years of age and wrote his book with that idea in mind. He stressed development of reasoning ability and led the pupil to formulate his own rules for solving problems. He also put in a great number of drill problems to give facility in performing fundamental operations. Examples were taken from situations familiar to children and only a few commercial problems were included.

The value of Colburn's work becomes evident when comparisons are made between the methods he advocated and prevailing practices before his time. Monroe points out that in Colburn's book topics are introduced by practical problems and the direction of progress is from the concrete to the abstract.[33] In the ciphering-book era, pupils were taught the abstract principle first and were then themselves expected to discover its application to concrete problems. Instruction was individualized and an instructor was a taskmaster who "set sums" but apparently felt no obligation to aid his pupils to overcome difficulties, until Colburn departed from the accepted practice of the time to provide both for class instruction and for a pupil to receive assistance from his teacher. The startling modernity of some of Colburn's ideas is shown by Monroe when he comments upon Colburn's discussion as to how a teacher should help a pupil:

> It is clear that Colburn understood that a difficulty initiates reflective thought. The pupil is first to meet a difficulty, feel a need, have a problem. This is the first step. Second, the pupil is to make his own hypothesis; the teacher is to keep her hands off. Unless the problem is one for which the pupil is not prepared, he will invent a way to solve the problem. It may be a crude one, but nevertheless a method that will control the value. The

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[32] Quoted from *ibid.*, p. 63.

[33] *Ibid.*, pp. 80–82.

thought process here is that of making hypotheses and verifying them. The instructor is in the background. Colburn would have his function be that of explaining to the pupil the meaning of the problem and its demands and to see that the pupil was finally made acquainted with the best method of solving the problem.[34]

To summarize, Colburn sought to impart the ability to make reasoned and skilful *use* of the processes of arithmetic. He envisaged a school in charge of a *teacher* who assisted his pupils to learn, not a *master* who "kept" school and listened to "re-citations" of verbalisms.

As would be expected, the combined influence of Colburn and Pestalozzi revolutionized the teaching of mathematics in America. Instruction in arithmetic was pushed down in the grades until it was taught to beginners, and the time given to arithmetic was increased until it came to dominate instruction much as reading had done previously.[35] Increasing emphasis was placed on skill and thoroughness in performing calculations.

GROWTH OF FORMALISM IN MATHEMATICS TEACHING

Unfortunately, the impulse to educational progress which resulted in the development of free public elementary schools, the beginning of the high-school movement, and the application of Pestalozzian reforms to the school program was by 1860 largely spent. In the case of mathematics, particularly arithmetic, teaching became formalized, partly because, as Monroe suggests, Colburn had influenced the making of arithmetic textbooks much more than he influenced the teaching of the subject, and in part because of overemphasis on disciplinary values. Reports of examining committees, as shown by the following quotations, portray the state into which instruction in arithmetic had fallen:

> Arithmetic has thus become a science of difficult trifles and intricate fooleries peculiar to common schools and remarkable chiefly for sterility and ill-adaptedness to any useful purpose.

[34] *Ibid.*, p. 85.

[35] *Ibid.*, pp. 92–93.

> In many districts the main thing in arithmetic is the definitions. In one school the first class was questioned as follows, "Spell arithmetic." "What is arithmetic?" "What is Roman Notation?" "What is a figure?" and so on during the recitation periods. The definitions must be word for word as in the book.
>
> The answer to the question, "How is a fraction expressed?" was given "by writing one number above the other." This was immediately corrected by the teacher to "by placing one number above the other."[36]

However, as Monroe suggests, even this tedious drill on factual information was better than the instruction of the ciphering-book period. Here is evidence of an attempt to teach—to develop an understanding of principles—which was wholly lacking during that time.

Nor was there lack of protest at the excessive formalization of instruction. In 1867 John D. Philbrick, superintendent of the Boston Public Schools, said:

> No exercises, no modes of preparing or conducting recitations, no explanations, should be required or allowed merely for the purpose of intellectual discipline; for it is safe to assume that the method of proceeding which is best calculated to communicate a competent knowledge of the subject will really be the best as a disciplinary process. Why make arithmetic hard for the sake of mere discipline, and then have no time left for algebra, geometry, or natural philosophy? . . . The proper use of an arithmetic textbook is to relieve the teacher, not wholly, but to a certain extent, from the task of preparing suitable problems for illustrating arithmetical principles and operations. . . . The hearing of recitations in arithmetic should be the exception, while teaching exercises should be the rule.[37]

Emphasis upon the disciplinary values of mathematics was especially strong at the secondary level. Francis Gardner, in 1867 headmaster of the Boston Latin School, apparently expressed a common philosophy of education when he said, "It is not what a boy learns at school that makes a man but *how* he learns it. All the knowledge that a faithful student acquires before arriving at manhood is as nothing compared with the intellectual strength he has gained and the ability he has of taking hold of any work that may present itself and doing it."[38]

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[36] *Ibid.*, p. 127.

[37] Quoted from Henry Barnard, *The American Journal of Education*, 19: 479 (1869).

The Reverend Henry F. Harrington, superintendent of public schools at New Bedford, Massachusetts, in an address before the State Teachers Association of Massachusetts in 1867 complained, "Arithmetic and grammar now engross most of the working hours of our schools, not so much for their intrinsic value as for the vicarious part they are expected to perform in disciplining the mind. The mathematics, being an exact study, has the credit of training the reasoning powers better than any other branch, and grammar is held in special honor, on the ground that the study of structure of language best disciplines the memory and judgment."[39]

The courses of study prescribed for the high schools of a number of American cities in 1867 are available in Barnard's *Journal.*[40] An analysis of these courses of study indicates that mathematical curriculums were essentially the same in subject-matter content as the mathematics curriculum of Phillips Exeter Academy fifty years earlier. In college-preparatory courses in 1867 algebra and geometry were still the only required mathematics courses. Four years of mathematics were offered, in most cases, for students who did not expect to enter college. Surveying and navigation still found a place in the curriculums of most schools, and in the high school of St. Louis, Missouri, analytical geometry and calculus were taught to students who expected to terminate their formal education at the secondary level.

Not only had teaching become formalized between 1818 and 1867, but the aim of mathematics education had also changed. Available evidence seems to indicate that in 1867 mathematics was taught for its disciplinary value, not for immediate utility. Apparently, the utilitarian subject matter of post-Revolutionary War days had been converted into an instrument for intellectual discipline through a

[38] *Ibid.*, p. 491. [39] *Ibid.*, p. 501. [40] *Ibid.*, pp. 469–576.

reversal of the process by which the practical Romans adapted the subject matter of the idealistic Greek education to utilitarian aims.

PROTESTS AGAINST EXCESSIVE FORMALISM

As we have seen, it was not without protest that mathematics, taught for its disciplinary value, had come to dominate American education. Protests were mainly of three types: (1) that mathematics should be taught for its practical utility; (2) that any subject well taught has disciplinary values; and (3) that the study of mathematics has no disciplinary values. The first type of protest was voiced by Superintendent Philbrick of Boston, who maintained that instruction in arithmetic "should have for its main object to communicate such a practical knowledge of numerical operations as would be most generally useful to the mass of people, without regard to particular occupations or pursuits."[41]

The second type of objection to the then current program of mathematics education was voiced by the Reverend Mr. Harrington, who argued that any subject "well and systematically taught" had disciplinary value.

The outstanding individual identified with the third type of protest against the domination of education by mathematics was the Scotch philosopher, Sir William Hamilton. In 1835 Dr. William Whewell, an English educator, published a treatise entitled *Thoughts on the Study of Mathematics as Part of a Liberal Education,* in which he asserted "the supremacy of mathematical study in the cultivation of the reasoning faculty over the classics or natural science, and as a useful gymnastic of the mind, far superior to logic itself."[42] In answer to Whewell, Hamilton published his famous attack on the educational values of mathematics in the *Edinburgh Review* for January, 1836. He said, in part, "From this general contrast, it will easily be seen, how an excessive study of the mathematical sciences not only does not prepare, but absolutely *incapacitates the mind,* for those intellectual energies which philosophy and

[41] *Ibid.*, p. 478.

[42] Barnard's *Journal,* 23:461 (1872).

life require. We are thus disqualified for *observation, either internal or external,*—for *abstraction* and *generalization,*—and for *common reasoning;* nay disposed to the alternative of blind *credulity* or of irrational *scepticism.*" (Italics in the original.)[43]

The study of mathematics, according to Hamilton, has no disciplinary value because "to minds of any talent, mathematics are [sic] *only difficult because they are too easy.*" (Italics in the original.)[44] The validity of the evidence which Hamilton offered in support of his contentions need not concern us here. It is sufficient to recognize that in the contentions of Hamilton, of Superintendent Philbrick, and of the Reverend Mr. Harrington, we find summarized the major arguments for decreasing the number of courses offered in mathematics.

In an earlier chapter it was shown that the dispute over the place of mathematics in secondary education has led the contending factions to accept *usefulness* as a criterion for the choice of subject matter but that they entertain widely divergent conceptions of what constitutes usefulness. The claim that study of mathematics has disciplinary values had its origin in attempts to establish educational values independent of the practical applications of mathematics; it was an attempt to show that the study of theoretical mathematics is *useful.* The contention of the Reverend Mr. Harrington, and of those who have since echoed his argument, that any subject properly taught has disciplinary value, is, in effect, an attempt to establish the usefulness of other subjects by claiming for them the virtues ascribed to mathematics. On the other hand, Sir William Hamilton and those who have followed in his footsteps deny that the study of mathematics has disciplinary values and therefore deny usefulness other than that derived from its practical applications.

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[43] Sir William Hamilton, Bart., "On the Study of Mathematics as an Exercise of Mind," *Discussions on Philosophy and Literature, Education and University Reform,* London, Wm. Blackwood and Sons, 1846, p. 279. First published in *The Edinburgh Review,* lxii:409–455. Quoted by Henry Barnard, *The American Journal of Education,* 23:463 (1872).

[44] Hamilton, *op. cit.,* p. 285.

THORNDIKE AND TRANSFER OF TRAINING

To claim that any subject has disciplinary value is to assume the possibility of transfer of training. If transfer of training takes place, it should be possible to demonstrate the fact experimentally. William James and others, notably Edward L. Thorndike, acted upon this hypothesis. One of the most significant of the experimental studies of transfer of training was made by Thorndike in 1922–1923. By analysis of the results of tests given to more than eight thousand children, Thorndike sought to determine the extent to which study of the various high-school subjects affected the general intellectual ability of pupils as shown by scores made on group intelligence tests. He reported:

> It should be noted that the facts . . . are not in opposition to the theory that general improvement of intellect results from school studies, and that studies vary in their value in this respect. The question is one of quantity. The traditional theory was (1) that the amount of general improvement due to studies was large; and (2) that the differences between studies in respect of it were large, so that the value of a study as training for the intellect was of very great importance relative to its value as special training in mathematical or linguistic or spatial or other thinking; and that (3) mathematics and languages gave much more of this general improvement than other studies did. The facts of Table XVI [not reproduced here], if corroborated by similar experiments, prove that the amount of general improvement due to studies is small; that the differences between studies in respect of it are small, so that the value of studies may be decided largely by consideration of the special training which they give. . . .
>
> By any reasonable interpretation of the results, it appears that the physical sciences are equal, if not superior, to languages and mathematics as now taught in our American cities, in respect of mental discipline from any point of view. . . . Something must probably be conceded to the more rigid and systematic and abstract nature of the thinking in mathematics, German, and physics as compared with that in English and History. . . . The expectation of any large differences in general improvement of the mind from one study rather than another seems doomed to disappointment. . . . After positive correlation of gain with initial abil-

ity is allowed for, the balance in favor of any study is certainly not large. Disciplinary values may be real and deserve weight in the curriculum, but the weights should be reasonable.[45]

Hence Thorndike, although he does not deny the possibility of transfer, denies that mathematics has any claim to special consideration. The dispute over the place of mathematics in present-day education is, in large part, due to conflicts of opinion concerning the type and amount of curriculum revision made necessary by this and other studies of a similar nature.

SUMMARY

It is readily evident, therefore, that the conflict of opinion concerning the nature and educational value of mathematics is not of recent origin. There seems to be no doubt that mathematics owes its origin to the human need for counting and measurement. Neither can one reasonably question that man's intellectual curiosity later led him to formulate principles governing quantitative relationships and so theoretical mathematics was born. In the subsequent development of mathematical thought there have been alternating periods, some in which the practical phase of mathematics was stressed and others in which the theoretical received major consideration. The virtues and defects inherent in any theory of education which stresses one phase of mathematics to the exclusion of the other will affect the results of instruction today as surely as ever they did in the past.

The dawn of written history found the Egyptians and the Babylonians making use of a rather extensive knowledge of mathematics. How they acquired this knowledge is not known, but apparently they took the utilitarian view of the subject and progress was exceedingly slow. The Greeks were interested in the theoretical aspects of mathematics and neglected the practical almost completely. This period of emphasis on the theoretical was one of rapid mathematical progress. The Romans were interested only in practical applications and failed to make significant contributions to the subject. There was stagnation throughout the Middle Ages, when the utilitarian point of view prevailed, and there was progress during the Renaissance, when men again became interested in knowledge for its own sake.

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[45] E. L. Thorndike, "Mental Discipline in High School Studies," *Journal of Educational Psychology*, 15:1–22 and 83–98 (January and February, 1924). Material quoted from pp. 96–98.

Periods of great productivity in the development of mathematical theory have been periods when the subject was studied for its own sake and not for the sake of possible practical applications. Periods of stagnation have occurred when emphasis was on the practical, utilitarian values of the subject. Abstract knowledge may be of potential value to the race in its struggle for better things; but the social value of knowledge comes from application. The practical Romans, who built bridges, roads, aqueducts, and sewers, made mathematics of more immediate value than did the Greeks with their highly intellectual but abstract theories.

It seems obvious that neither the practical-utilitarian phase of mathematics nor the theoretical-cultural phase, either taken by itself, is satisfactory as a basis on which to decide the place of mathematics in a desirable program of education. These two phases are not independent of one another. They are mutually dependent; supplementary parts of a complex whole.

Mathematics apparently came into the curriculum of American schools to fill a practical, utilitarian need. There was a demand for the arithmetic of shop and market place, for the mathematics of the surveyor and the navigator. It was only when educators sought to use the theory of formal discipline to justify obsolete methods and archaic subject matter that the educative value of mathematics began to be questioned.

The work of Colburn is evidence of the possibilities for educational progress inherent in an effective synthesis of the two phases of mathematics. A satisfactory program for mathematics education in the United States must take both phases of the subject into consideration.

Possibly the most significant cause of unrest, contention, and controversy over the proper place and function of mathematics in American education is that civilization is now passing through one of those periods of turmoil and change which, through the ages, have marked a transition from one culture epoch to another. Technological advances have so increased the interdependence of nations and peoples that the present period of conflict and confusion promises to be even more far-reaching in its effects than was the Reformation or the epoch which started with our Revolutionary War. The present struggle has been characterized as a struggle for economic democracy. May it prove as fruitful of educational advance as similar periods in the past.

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Chapter 4

Educational Theories and the Teaching of Mathematics

THE DOCTRINE OF SPECIFIC OBJECTIVES

THE first attempt to meet the vexing problems created by the unprecedented growth in secondary-school enrollments in the United States during the early years of the present century took the form of a movement to make education more practical. If the old concepts of faculty psychology were no longer valid, if there was no automatic and extensive transfer of training, then those studies which could find the readiest and most complete application to the problems of everyday life would be of most value to a student. If, as Thorndike insisted, transfer of training from one situation to another took place when, only when, and to the extent that the two situations involve identical elements, then the objectives of any valid educational program would necessarily be specific, not general. It would be necessary to devise some means of finding what specific skills, attitudes, and bits of information would be needed by each individual. These specific learnings would then become the basis for the curriculum.

CURRICULUM BUILDING BY ACTIVITY ANALYSIS

The starting point in determining what should be included in the curriculum, according to Franklin Bobbitt who was one of the outstanding leaders in education in the period just following World War I, was an analysis of the activities in which people engage. Bobbitt wrote, "When we know what men and women ought to do, along the many lines and levels of human experience, then we shall have before us the things for which they should be trained. The first task is to discover the activities which ought to make up the lives of men and women; and along with these, the abilities and personal qualities necessary for proper performance. These are the educational objectives."[1]

Obviously the assumption that the aim of education is preparation for adult life was inherent in this specific-objectives approach to curriculum construction. Bobbitt insisted that "Education is primarily for adult life, not for child life. Its fundamental responsibility is to prepare for the fifty years of adulthood, not for the twenty years of childhood and youth."[2] However, he was careful to point out that education could not be concerned with adult activities to the exclusion of the activities which make up the life of a child. He said, "As a matter of fact, however, life is lived at all ages. It ought to be a good high type at all ages. . . . Education is not aiming at a high quality of behavior at any one level, whether infancy, childhood, youth or adulthood. . . . It is aiming at the *life series of activities. The objectives of education are all the activities which ought to make up the totality of human life from birth to death.*" (Italics in the original.)[3]

According to Bobbitt, therefore, although it was essential that a curriculum maker know what activities were typical of children at any age level, his ultimate concern was with adult activities. For, Bobbitt argued, "It is here that civilization manifests itself in its highest terms. It is the activities of the mature level which are the ultimates. These are the goals in terms of which we can see and evaluate the earlier portion of the life series. The first step of activity analysis, therefore, is to discover what civilized activities are at their highest at the mature or adult level."[4] In other words, Bobbitt would analyze the activities of children at various levels merely to determine how best to utilize these activities in preparing for adulthood. The activities of adults would, however, determine the objectives of education.

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[1] Franklin Bobbitt, *How to Make a Curriculum*, Boston, Houghton Mifflin Co., 1924, p. 9.

[2] *Ibid.*, p. 8.

[3] Franklin Bobbitt, *Curriculum Investigations*, Supplementary Educational Monographs No. 31, Chicago, University of Chicago, June, 1924, p. 4.

These objectives of education would be quite specific. Bobbitt said, "The objectives should be stated in definite terms. . . . General unanalyzed objectives are to be avoided. For the ten major divisions of human action it would be possible to state ten corresponding abilities. These would be so general as to be practically useless for curriculum making. 'Ability to care for one's health,' for example, is too general to be useful. It must be reduced to particularity: ability to manage the ventilation of one's sleeping room, ability to protect one's self against micro-organisms, ability to care for the teeth, and so on."[5]

Bobbitt listed several hundred "major objectives of education." Apparently he was of the opinion, as was Morrison, that "by a process of reasoned analysis and experimental verification in the classroom," he could make education approach the status of an exact science.[6] Presumably, to accomplish this end it was only necessary to determine all the activities in which a given individual was likely to engage and then to devise teaching exercises which would provide adequate training in those activities.

Wilson's study of the arithmetic needed by adults affords an example of the activity-analysis approach to curriculum making. He collected the actual problems which adults were called upon to solve in the course of their everyday activities. The survey covered twenty-three cities and certain rural areas. During a period of two weeks a group of 4068 adults representing 155 different occupations reported solving 14,583 problems involving 21,898 different arithmetic operations. Wilson then determined which of the processes commonly taught were used most and which but seldom or not at all.[7] On the basis of his analysis, Wilson recommended that certain topics such as addition and multiplication of numbers under one hundred—topics having a relatively high frequency of occurrence—be given sufficient attention that mastery by practically all pupils will be assured. On the other hand, he said that the time devoted to topics which are used but seldom or not at all could be materially reduced. According to him, topics such as apothecaries' weights, partial payments, square root, decimals, compound interest, proportion, and much of mensuration—to name only a few—were used so seldom that the time formerly devoted to their study could be spent more profitably on topics having a greater frequency of use.

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[4] *Ibid.*, p. 4.

[5] Bobbitt, *How to Make a Curriculum*, p. 32.

[6] Henry Clinton Morrison, *The Practice of Teaching in the Secondary School*, Chicago, University of Chicago Press, 1931, p. 26.

UTILITY AS CRITERION FOR CHOICE OF SUBJECT MATTER

Wilson assumed, in making these recommendations, that the chief educative value of arithmetic is its utility in the affairs of everyday living. He said, "While not denying the cultural and disciplinary value of arithmetic—in common with any subject systematically studied and well taught—it is assumed that arithmetic in the grades is justified only on the basis of its utility in the common affairs of life."[8] Charters also argued that usefulness should be the basis for determining the objectives of education. He said, "The standards of our day demand that our courses of study be derived from objectives which include both ideals and activities, that we should frankly accept *usefulness* as our aim rather than *comprehensive knowledge*, and that no fictitious emphasis should be placed upon the value of formal discipline." (Italics in the original.)[9]

But a given topic in mathematics may have great utility for one person and little or none for another depending on their vocations.

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[7] Guy M. Wilson, *What Arithmetic Shall We Teach?* Boston, Houghton Mifflin Co., 1926.

[8] *Ibid.*, p. 1.

[9] W. W. Charters, *Curriculum Construction*, New York, The Macmillan Company, 1929, p. 4.

Hence it is necessary to distinguish between vocational education and general education. Bobbitt defined general education as follows: "That portion of the training . . . which is of general need, whatever be one's occupation or station in life, we shall call . . . *general education,* the term specifying non-specialized education, or the common element in the training of all persons."[10]

Activity analysis will reveal a need for mathematics in many vocations, such as engineering, statistical analysis, and actuarial work. But does mathematics have a place in general education? Wilson's study of the uses of arithmetic, mentioned previously, was an attempt to answer this question in part. Bobbitt reported the results of an analysis of vocabulary—the terms man applies to his activities:

> One of the surprising things is the way in which the *quantitative* element stands out in the vocabulary used. In part, the terms were numbers and other words referring to exact quantity. In still larger measure, they were terms indicative of approximate quantity, such as "few," "several," "many," "much," "little," "large," "small," "abundant," and "meager." The evidence of language is that even in the prescientific age during which most of our language was evolved and most of the writings produced which were analyzed, the quantitative aspect of thought was an outstanding characteristic. Much more must it be the case as we enter an age ruled much more completely by exact science. It appears to be unequivocal evidence in support of a generous inclusion of the quantitative element in the curriculum.
>
> In the last statement we did not say "mathematics," since this term usually covers many things for which justification was not found. The facts appear to call for a type of mathematics for general education that is very different from the usual type, which, apart from its alleged purposes of formal discipline, has been organized mostly for the use of specialists.[11]

The distinction which Bobbitt made between mathematics and what he termed "the quantitative element in the curriculum" was, in reality, only the age-old distinction between the practical and the theoretical. He said that mathematics was not used in life and that the abstract need not be studied.[12] Evidently he assumed that the educational significance of mathematics is derived entirely from the number-quantity-and-measurement phase of the subject. He said, "The quantitative aspect of things is universal. Everything that has existence exists in some measure, degree, quantity or number. To see things with exactness, one must, among other things, see them quantitatively. To deal with them practically, one is concerned with quantities, numbers, measures."[13]

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[10] Bobbitt, *How to Make a Curriculum,* p. 66.

[11] Bobbitt, *Curriculum Investigations,* p. 38.

EDUCATIONAL UTILITY OF THE CULTURAL ASPECTS OF MATHEMATICS

As we see it, the "quantitative element in the curriculum" may have cultural values as well as practical usefulness. Minnick distinguishes between practical and cultural values of a subject as follows:

> A subject may be said to have a practical value when it has specific applications. Mathematics has a practical value to a banker when he solves financial problems involved in his business; to the carpenter when he cuts two pieces of timber to fit at right angles; or to the statistician when he computes a correlation. . . . A subject may be said to have cultural value when it has general usefulness. If a subject enables a person to understand his social and physical environment, to converse more fluently and intelligently about various topics, to meet various situations better, if it satisfies his desire to know or brings him artistic satisfaction, it has for him cultural value.[14]

If Minnick's distinction between practical and cultural be accepted, Bobbitt should consider the cultural values of mathematics to be of greater educational significance than the practical value. For example, he includes in his monograph on *Curriculum Investigations* a chapter on "The Mathematics Used in Popular Science," written by Robert Scarf, which begins: "Mathematics is an instrument of human thought. To find the mathematics which should enter into the curriculum of general education, let us find that mathematics which persons in general actually use in their thinking. The thought of most persons is at its highest as they read the printed thought of the world's intellectual leaders. Let us find, therefore, the mathematics met with in this reading, and we shall find the mathematical tools which intelligent persons need to use and for the use of which they should be sufficiently trained."[15]

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[12] Bobbitt, *How to Make a Curriculum*, pp. 158–164.

[13] *Ibid.*, p. 146.

[14] J. H. Minnick, *Teaching Mathematics in the Secondary Schools*, New York: Prentice-Hall, Inc., 1939, pp. 38–41.

Scarf suggests that, since science is the field of thought which involves the greatest degree of accuracy, the mathematics needed in reading scientific matter will probably be the best indication of the range of mathematics needed in general education. Hence, he reports on the results of an analysis of science articles in five general magazines and three books on popular science. Some of his conclusions concerning arithmetic are as follows:

> It is so obvious that the curriculum must develop the power to use numbers up to 1,000,000 that no complete count was made of the smaller numbers.
>
> One is called upon to read a large number of fractions. These cover a wide range. . . . [One finds] a considerable number of fractions with denominator greater than 1,000,000.
>
> Most of the decimals found are simple, but there is a sufficient range to indicate that the reader should know the meaning of any decimal expression.
>
> The examples of arithmetic problems given show that anyone who wants to read science should know enough about arithmetic so that he does not have to regard the computations as mysterious juggling with figures. . . . The problems involve all sorts of arithmetic—the four fundamental operations with whole numbers, decimals and fractions, powers and roots, ratio and proportion, geometric progression, finding the average, and telling time.[16]

The reader of popular science will need the descriptive and informational phases of algebra, Scarf finds. Mathematicians will not agree with his conclusion that "algebra consists chiefly of processes and skills and has very little content and a meager vocabulary."[17] Scarf finds much need of geometry for "form." There is also need, he says, for some knowledge of higher mathematics.

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[15] Bobbitt, *Curriculum Investigations*, p. 119.
[16] *Ibid.*, pp. 120–139.
[17] *Ibid.*, p. 142.

THE SPECIFIC-OBJECTIVES PROGRAM

Yet Bobbitt proposes to provide the training which Scarf's analysis shows to be necessary by elimination of the abstract. He says, "We are suggesting diminution of attention to quantity in the abstract; and greatly increased attention to quantity in the concrete. We do not see how this is to be accomplished practically for general education except to diminish the abstract mathematics in the mathematics department, and to increase the quantitative element in the handling of the so-called content subjects. Time saved from the former can be used for the latter."[18] Bobbitt outlines his suggested program as follows: "An understanding of the number system, concrete and general, and skill in performing the essential operations needs to be taken care of in the mathematics classes. This, however, ought not to require a great amount of time, since the bright can travel with speed and the dull need not go far. Then beyond the needed minimum of such abstract mathematics, the mastery can be left to the use of mathematics in the content studies. Except for the alphabet of the field, the way to learn mathematics is to use it."[19]

From all this it is readily evident that the doctrine of specific objectives, since it is based on the assumption that the aim of education is preparation for adult life, would lead to a predetermined program which the learner would be expected to master rather completely. In setting up such a program, the curriculum maker would proceed on the assumption that mathematics is the science of number, quantity, and measurement, and would deny that it has any educative values other than those of practical utility. Obviously, such a program of education would pay little attention to the theoretical aspects of mathematics. In view of the experience of the past, as recorded in the history of mathematics and of education, one would expect to find that this emphasis on the practical would fail to provide a basis for the continued progress of mathematics as a science or for new practical applications of mathematical theory. A single example will serve to show the consequences of this weakness in the specific-objectives program.

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[18] Bobbitt, *How to Make a Curriculum,* p. 154.
[19] *Ibid.*, pp. 154–156.

Wilson concluded, as a consequence of his analysis of the topics of arithmetic used by adults, that compound interest is one of the topics of little practical use to most individuals. Yet most investments and all insurance and installment-buying transactions involve compound interest. Quite probably the people who reported their uses of arithmetic in his survey did not solve any problems in compound interest, but that does not mean that compound-interest problems did not arise. It may simply mean that the need for computing compound interest was not recognized or that prepared tables were consulted. Probably the salesman looked up the amount of the payments in a table and the prospective installment buyer did not attempt to calculate the rate of interest he was to pay. Or perhaps Wilson's people made no investments, bought no insurance. Wilson made his study in 1918, when installment buying was probably not as widespread as it is now. In any case, the activity-analysis method fails to anticipate need for new mathematical theory. Other instances of the same kind might be cited but the foregoing is sufficient to indicate the justice of Bode's criticism that the specific-objectives approach to curriculum construction is not compatible with democratic education. Bode says:

> The significance of the democratic movement lies in the notion that training for specific objectives cannot be the whole aim of education for the reason that the purpose of this movement is precisely to make over the social order and our present modes of living so that we may progressively substitute new objectives for old ones. Any scheme of education that fails to make provision for this element of progress is, so far forth, hostile to the democratic purpose of humanizing both education and life. An educational ideal which is content to train pupils for predetermined specific objectives is better suited to a static than to a dynamic social order.[20]

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[20] Boyd H. Bode, *Modern Educational Theories*, New York, The Macmillan Company, 1927, p. 79.

Bobbitt gives as one of the guiding principles and assumptions which are to aid in the determination of the program in mathematics education the statement that a "consumer" of mathematics need not know the principles on which a given technique is based. There is a classic example from the field of statistics which will serve to demonstrate the falsity of this assumption. The Pearsonian coefficient of correlation is a statistical measure of the relationship which exists between two variables. If we obtain data representing the quantity of alcoholic liquors consumed in the United States for a series of years and similar data concerning the total salaries paid schoolteachers during the same period and then compute the coefficient of correlation between these two variables, we shall find, surprisingly enough, that the result indicates almost perfect correlation between the two. Obviously, there is no direct connection between the amount of whiskey consumed and salaries of teachers. The apparently close relation grows out of the fact that the quantity of liquor consumed in a year as well as the salaries paid schoolteachers will both depend upon general economic conditions. Yet the "consumer" of mathematics who knows nothing of the theory of statistics is likely to be misled by the high correlation into believing that a close relation exists between whiskey drinking and teachers' salaries. It is precisely this innocence of the assumptions basic to statistical formulas and of the real meanings of statistical measures which is responsible for the misuse of statistics in education, as mentioned by Walker[21] and Orata.[22]

The contradictory results reported by Wilson and Scarf point to a fatal weakness in the whole theory of activity analysis. Modern civilization is so complex and the activities of people are so diverse that merely to analyze these activities and to tabulate the specific elements is of little value to a curriculum maker as a means of indicating what should be included in an educational program. Analysis must have reference to particular educational aims. For example, it is a comparatively simple matter to make an analysis of the process of laying bricks. But the educational value of the analysis depends upon the use to be made of the information. If we assume that a bricklayer is eager to learn methods which will enable him to lay bricks more rapidly, then the analysis of his activities will be a systematic attempt to discover unnecessary movements which may be eliminated with a resultant increase in speed and efficiency. But the bricklayer may not want to lay bricks faster. He may want to make the job last as long as possible and at the same time give the impression that he is working at maximum rate. In that case, the analysis of his activities will be directed toward discovery of ways of slowing down the work. In other words, activity analysis must be directed toward some predetermined end. Wilson and Scarf obtained contradictory results because they had different ends in view.

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[21] Helen M. Walker, "Concerning the Standard Error of a Difference," *Journal of Educational Psychology*, 20:53–60 (January, 1929). See also "Problems in Training Research Workers," *Journal of Educational Research*, 26:442–448 (February, 1933), by the same author.

[22] Pedro T. Orata, "Insignificance of the Statistically Significant," *Educational Administration and Supervision*, 20:321–340 (May, 1934).

Bobbitt's theories about curriculum making were based on the assumption that what people do is what they ought to do. Unfortunately, this ideal is not achieved in practice. Furthermore, the fact that most people engage in certain activities does not necessarily indicate that these activities have great educational significance. It might indicate that, since people are likely to learn them without special training in school, these activities need not be considered in the educational program.

TEACHING METHODS

The procedures to be used in teaching a curriculum derived by the process of activity analysis were summarized by Morrison in his so-called "mastery formula": "Pretest, teach, test the result, adapt procedure, teach and test again to the point of actual learning."[23] Morrison assumes that whatever subject matter is chosen should be divided into specific learning units each of which presents "a comprehensive and significant aspect of the environment, of an organized science, of an art, or of conduct, which being learned results in an adaptation of personality."[24] By "adaptation" Morrison means a change in the personality of a learner. It implies a modification of attitude, of outlook, or of skill which results in the better adjustment of the learner to his environment.

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[23] Morrison, *op. cit.*, p. 81.

This concept of adaptation is basic to Morrison's—and perhaps the whole specific-objectives—theory of education. An adaptation is, according to Morrison, a unitary and permanent thing. By unitary he means that there are no degrees of adaptation. Either the individual has made the adaptation or he has not. By permanent Morrison means that an adaptation once made is never lost. It may, however, be transformed through new adaptations or "through the rise of pathological inhibitions."[25]

The subject matter of education, in terms of the adaptation concept, becomes a series of units each of which, when mastered, results in a specific change in the personality of the individual. The method of activity analysis will be used to determine the specific attitudes, skills, and bits of information that are needed and desirable. Then the teacher will apply the mastery formula until the learner has made the required adaptations. Education is, therefore, merely a matter of making a finite number of changes in the personality of a student. Hence, according to this theory, through the use of suitable measuring techniques education can be made an exact science.

Specific procedures used in seeking mastery of a given unit will, of course, vary according to the nature of the subject matter concerned. Mathematical materials would constitute what Morrison terms a science-type unit. Application of the mastery formula to a science-type unit involves five steps, which are termed by Morrison exploration, presentation, assimilation, organization, and recitation. Exploration begins with a pretest, either oral or written, followed by class discussion. In effect, it determines how many of the desired learnings that are expected to be outcomes of that particular unit a student has already acquired. The second step, as applied to mathematical subject matter, is simply one of presenting a new concept or process. If, for example, the unit being studied is on the quadratic formula, the teacher will develop the formula and show how it might be used. The third step, assimilation, is the process by which a student "makes the new understanding his own." It is the step which ends with mastery of the materials to be learned. The fourth step, organization, consists in arranging the materials of the unit into logical, coherent form. In mathematics this step is not needed, according to Morrison, since the materials will already be organized in logical order. In the final step, recitation, each pupil demonstrates his mastery of the unit by presenting the material to teacher and class.

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[24] *Ibid.*, pp. 24–25. [25] *Ibid.*, pp. 21–22.

According to Breslich, the essential characteristics of a unit in mathematics are: "(1) It must be a compact body of closely related facts and principles. (2) It must be possible to present the unit as a whole in concise form, giving the learner a clear general conception of it. (3) It must be susceptible to testing for complete understanding."[26] A unit must, therefore, be comprehensive and significant. It must present propositions such that understanding these propositions will cause modifications in the behavior of a learner. The scope of a unit must be sufficient to lead to the desired adaptation—modification of behavior.

Morrison suggests that a unit on the equation in algebra may be taken as typical of units in mathematics. Concerning the nature of such a unit he says, "We are not concerned with all kinds of equations. If we were we should have a chapter and not a unit. We are not concerned with possible applications of the equation. . . . We are not concerned with ideational content to which the equation may be applied but which is at present far beyond the experience of the pupils. We are not concerned with an expression in the form of an equation in which the sign of equality is a mere substitute for the verb 'to be.' We are concerned with the nature of an equation as an instrument in mathematical thinking and with the fundamental manipulations."[27] A unit on equations, therefore, must present sufficient material, and only the material needed, to insure mastery of the idea that an equation is a statement of equality and that certain specified operations may, under certain conditions, be performed upon the members of that equality.

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[26] Ernst R. Breslich, *The Teaching of Mathematics in the Secondary School*, Chicago, University of Chicago Press, 1930, Vol. I, p. 145.

EVALUATION OF SPECIFIC-OBJECTIVES PROPOSALS

The sheer futility of attempting to isolate all the specific ideas, skills, and bits of information which any given individual will need to possess in order to be competent to deal with the complex world of today is at once apparent. Yet acceptance of the specific-objectives concept of education would require that this feat be attempted together with the even more impossible tasks of organizing the materials so selected into specific teaching units and then obtaining mastery of all these units by all pupils.

Moreover, even if these educational labors of Hercules could be performed, they would not achieve the ends sought. An individual competent to cope with a changing world must be able to adjust to new situations, to recognize, analyze, and solve novel problems. The specific-objectives program does not give training in recognizing and solving new and significant problems. It merely proposes to teach specific solutions to specific problems already recognized and analyzed. It is preparation for a static world. Since thinking is, as Bayles says, a process of solving a problem, the specific-objectives program will not develop the ability to think.[28] It merely imparts information. Thinking and knowing are not the same thing. They are different processes. Bayles says, "Thinking may be thought of as coming to know. Knowing is the outcome. Knowing may be in the form either of particularized items (memory) or of generalized interpretations (understanding). Whether one form or the other, knowing should be recognized as different from thinking, and a teacher should never be misled into the belief that, because a student has been given knowledge in either form, he has also been given training in thinking."[29] As Bayles points out, Morrison has based his teaching program on just this error.

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[27] Morrison, *op. cit.*, p. 237.

[28] Ernest E. Bayles, *The Theory and Practice of Teaching*, New York, Harper & Brothers, 1950, p. 108.

In short, the doctrine of specific objectives is based on the assumptions that the aim of all education is preparation for adult life, that transfer of training is dependent entirely upon the presence of identical elements in the two situations, that mathematics is merely a matter of number, quantity, and measurement, and that learning to know is learning to think. It implies an impractical method of selecting subject matter and a teaching method based on the assumption that learning the solution to problems is the way to learn how to solve problems. Because a program of education based on the doctrine of specific objectives will not provide training in the thinking needed to solve novel problems, such a program will not provide education for changing conditions, hence will not provide adequate preparation for assuming the responsibilities of citizenship in a society which is designed to promote constant change.

MATHEMATICS AND THE PROGRAM OF THE PROGRESSIVES

A second proposal for reforming educational practices which was widely advocated during the years following the First World War was that advanced by the so-called "Progressive education" group—a group which has been characterized as "the protagonists of the child-centered school, the advocates of child freedom, activity, pupil initiative and responsibility, in short, the rebels and protestants against the regimentation of the formal school."[30] As might have been expected from a group of "rebels and protestants," there was not complete unanimity concerning anything about their program except protestation against the philosophy, curriculum, and methods of the traditional school. Proposals for the advancement of education by these "Progressives with a capital P" ranged all the way from

[29] *Ibid.*, p. 109.

[30] Harold O. Rugg and Ann Shumaker, *The Child-Centered School*, Yonkers, World Book Co., 1928, p. 33.

attempts to make the traditional curriculum function more effectively by means of devices such as "individualized instruction"[31] and the "socialized recitation" to proposals that the school become an active, policy-forming agency in "building a new social order."[32] A certain basic philosophy of education and certain characteristic concepts regarding the subject matter and methods of education ran through the proposals of this group, however, whether there was agreement as to detail or not.

PHILOSOPHICAL BASIS OF THE PROGRAM OF THE PROGRESSIVES

Philosophically, the "new education" of the Progressives had its basis in the ideas of John Locke and Jean Jacques Rousseau. Some writers contend that the philosophic roots of the child-centered, activity-type school go back to the ancient Greeks—to Plato and Aristotle.[33] Hence, the philosophy of the "new education" was no recent, mushroom growth but was instead an expression of an educational philosophy that is much older than free, secular, public education. And, as Kandel has pointed out, the conflict between the Progressives and the advocates of more formal education was more fundamental than an argument over whether instruction should center around subjects or grow out of the "felt needs" of pupils. Kandel said, "It is a conflict between idealism in one form or another and pragmatism, between classicism and romanticism, between those who believe that the experience of the race has something to contribute to the enlightenment of the individual in the modern world and those who stress the primacy of immediate experiencing by the individual in and through the environment in which he lives."[34]

Rugg and Shumaker saw this conflict as a battle between the doctrines of discipline and growth. They, like Kandel, saw it as a contemporary manifestation of an ancient dispute. They said, "This warfare is indeed but a recurrence on a more diversified and universalized scale of the ancient battle waged by educational reformers of earlier centuries—Rousseau, Pestalozzi, Herbart, and Froebel, to name a few conspicuous leaders. Educational systems have always reverberated with this conflict between adult and child life, between the logical and the psychological, between control and freedom, iron rule and spontaneity, but never until our own generation has there been such a vivid picturing of thought concerning the agelong argument."[35]

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[31] Carleton Washburne, "The Winnetka System," *Progressive Education*, 1: 11 ff. (April, 1924). See also the articles by Burk and Parkhurst on the Dalton plan.

[32] Geo. S. Counts, *Dare the Schools Build a New Social Order?* New York, The John Day Co., 1932.

[33] Rugg and Shumaker, *op. cit.*, pp. 166–167.

[34] Isaac L. Kandel, "Prejudice the Garden Toward Roses," *The American Scholar*, 8:79–82 (Winter, 1938–1939).

The conception of education as discipline has its philosophical basis in the doctrine of original sin—the idea that nature is inherently evil. The Progressive doctrine, on the other hand, assumed with Rousseau that everything in nature is good until it is corrupted by man. A system of education based on the thesis that nature is good will stress natural development of the individual. Methods and materials of instruction which permit closest approach to "natural situations" will be preferred. This system implies a large degree of individual freedom and initiative on the part of a pupil, elimination of predetermined curriculums, and attention focused on problems of immediate interest. In short, it implies the Progressive program.

The doctrine of original sin implies that nature is bad, that man's instincts must be curbed and repressed. Man is totally depraved and must be disciplined. This requirement implies large restriction of individual freedom, both physical and mental. The assumption is that if an individual is left to his own devices he will drift into evil ways. Therefore, he must be forced to do things against his will, on the theory that in so doing he will strengthen his power to combat weaknesses of the flesh. A sense of duty must be the controlling force in an individual's life. The doctrine of original sin was one of the tenets of the Puritan religion; Jonathan Edwards was expressing an attitude consistent with that doctrine when he characterized children as "young vipers and infinitely more hateful than vipers."[36] Such was the attitude held in schools founded to perpetuate that savage creed, schools that in turn were the forerunners of present-day public schools. Kilpatrick has in mind the vestigial remains of this doctrine when he says, "The old school mistrusted childhood, counting it totally depraved. Later and milder statements were that children are 'naturally lazy' or, if not lazy, at least 'normally and constitutionally indolent,' or perhaps, still more kindly, 'averse to continued effort along a given line.' A closer and fairer view shows that these evil imputations are better understood as adult failures, the failure and unwillingness to understand children as growing and treat them so."[37]

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[35] Rugg and Shumaker, *op. cit.*, p. 35.

An either-or philosophy necessitates choice between these two alternatives. The dispute as to which horn of the dilemma is to be preferred is probably as old as civilization. A third alternative, the conception that nature is neither good nor bad but neutral, is the basis for the educational philosophy of John Dewey.

Leaders of the disciplinary group, according to Rugg and Shumaker, are subject-matter specialists, public-school administrators, headmasters of private schools, and "even most of the professors of education and the protagonists of the scientific method."[38] To them education means "discipline, logical thinking, power of sustained intellectual effort, the retention of classified knowledge." On the other hand, according to Rugg and Shumaker, the Progressives—"proponents of freedom in education"—"have focused attention upon the continuous growth of the child, upon freedom, initiative, spontaneity, vivid self-expression."[39]

The experimental school organized by John Dewey at the University of Chicago in 1896 was one of the first schools in America to try the Progressive method. In this school Dewey attempted to give concrete expression to the beliefs that "the school cannot be a preparation for social life except as it reproduces the typical conditions of social life"; that "industrial activities are the most influential factors in determining the thought, the ideals, and the social organization of a people"; and that "the schools should be life, not a preparation for living."[40] Rugg and Shumaker characterize this school as "a thoroughly radical institution . . . with neither school subjects nor conventional furniture, the first important overt expression of the growing protest against the formal school."[41]

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[36] Quoted by E. P. Cubberley, *The History of Education,* Boston, Houghton Mifflin Co., 1920, p. 374.

[37] Lois Coffey Mossman, *The Activity Concept,* New York, The Macmillan Company, 1938, pp. xi–xii. From the Introduction by William H. Kilpatrick.

[38] Rugg and Shumaker, *op. cit.,* p. 35.

[39] *Ibid.*

But Dewey's greatest service to education has been rendered in the field of educational philosophy, not in experimental determination of method. The theories of Dewey have undoubtedly influenced the creed of Progressive education although it is sometimes difficult to see the connection between the doctrines of the master and the practices of some of his professed disciples.

ASSUMPTIONS BASIC TO THE PROGRAM OF THE PROGRESSIVES

Rugg and Shumaker comment on Dewey's statement, that education is "that reconstruction or reorganization of experience which adds to the meaning of experience and which increases ability to add to the course of subsequent experience," as follows: "For education in the Century of the Child aims at nothing less than the production of individuality through the integration of experience. The whole child is to be educated. Hence the materials of education are as broad as life itself. For experience is not only an intellectual matter, it is physical, rhythmic, emotional. Thus the vocabulary of the new school has coursing through it a unitary, integrating theme: individuality, personality, experience."[42]

Kilpatrick defines education in much the same way as Dewey, but uses the term "life" instead of "experience." However, Dewey defines what he means by "experience" whereas Kilpatrick does not define

[40] Cubberley, *op. cit.*, p. 781.
[41] Rugg and Shumaker, *op cit.*, p. 38.
[42] *Ibid.*, p. 5.

what he means by "life." The latter appears to include everything done while one is alive, whereas Dewey's term "experience" refers only to those activities performed with anticipation of the consequences of the performance. Kilpatrick says:

> Education, then, is desirably such a process of living as remakes life. Remakes it not once nor occasionally at long intervals, but if possible continuously remakes it. So that each learning experience leaves the learner at once with a broader outlook, at once more disposed and better equipped to go on to further like fruitful experiences. This is what we mean by saying that education is such a process of associated living as continuously remakes life, carrying it always to higher and richer levels, not only for the individual but also for all whom he influences. . . . Education is the continuous reconstruction of life to ever higher and richer levels. This is the definition of education for the new curriculum.[43]

The educational theory growing out of the conception of education as a continuous remaking of life would be expected to be a tentative, growing, evolving body of doctrine, but certain basic tenets would be expected to be rather plainly evident. Rugg and Shumaker say, "Haltingly the new philosophy is being evolved; bit by bit a mosaic of theory is being constructed, mostly of excerpts culled from the educational writings of Dewey and his followers. Gradually basic articles of faith are emerging to which all these new schools subscribe. For back of the varying, overt manifestations of the new spirit stand a few fundamental beliefs, a few distinctly new conceptions of growth, experience, the course of learning and of education."[44]

Freedom is listed as the first of these articles of faith, "freedom to develop naturally, to be spontaneous, unaffected, and unselfconscious."[45] This implies freedom to move about—the disappearance of the old formal schoolroom with fixed furniture, set time schedule, teacher-imposed tasks, and repression of activity. The new schools build on the activity, the spontaneous expression of ideas, and the informality which the traditional school repressed.

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[43] William H. Kilpatrick, "Statement of Position," *Twenty-Sixth Yearbook of the National Society for the Study of Education, Part II*, Public School Publishing Co., Bloomington, Illinois, 1926, p. 131.

[44] Rugg and Shumaker, *op. cit.*, p. 55.

[45] *Ibid.*, pp. 55–56.

A second article of faith is that responsibility and initiative for planning and administering the program rest with the pupils. Thus Rugg and Shumaker say, "These schools believe that boys and girls should share in their own government, in the planning of the program, in the administering of the curriculum, in conducting the life of the school. In the elementary division of some of these schools, during an informal morning discussion period, children, with the teacher as a wise but inconspicuous member of the group, consider together what they are to undertake during the day. The routine needs of the school as well as the lesson assignments, the planning of excursions and exhibits, and the criticism of reports are taken over by the pupils."[46]

Kilpatrick sums up his discussion of "purposeful activity" by saying:

> The four steps in a typical instance of purposeful activity are purposing, planning, executing, and judging. The last includes two kinds of judging, the specific as to the success of the result and the generalizing as to what general lessons have been learned. These steps mutually imply each other, but may in particular instances go several at a time. For learning to go on best, the learner should himself take each step in the process. . . . If a child is about to fail with any step, the teacher may properly intervene to save him from failure; for final failure may mean discouragement and lessened learning. But the teacher will help by helping the child to help himself.[47]

Thus the program is determined largely by the children; learning becomes an active, pupil-purposed process rather than a passive, adult-purposed one. This is, say the Progressives, putting the true theory of democracy into school practices. They argue that only by practicing democratic self-government in school can children learn to be effective citizens in a democracy.

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[46] *Ibid.*, p. 57.

[47] William H. Kilpatrick, *Foundations of Method*, New York, The Macmillan Company, 1925, pp. 215–216.

A third article of the Progressive faith is that the basis of all learning is in reacting to a situation; the Progressive school is an activity school. According to Rugg and Shumaker, "The newer education regards the active child as the truly growing child. Not activity for activity's sake—energy exploding in random movements—but activity which is a growing toward something more mature, a changing for the better. The true criterion of educative activity is prolonged attention and concentrated effort. Such then is the activity which the new education writes into its articles of faith."[48]

In the new education, the program of the school is organized about the interests of children, as contrasted with the traditional school, which organized its program about school subjects. If a traditional school subject happens to coincide with some center of child interest, as in the case of reading, then it is retained, but if an old school subject does not "coincide with life interests, either of children or adults," it is dropped. "This new plan of organizing the curriculum around units of pupil activity gives greater promise of widespreading, educational achievement for the pupil than does the dry, intellectualized, logical arrangement of subjects-to-be-learned of the old school. It is vitalized by interesting and purposeful activity that has an intimate connection with the child's personal life."[49]

This fourth article of faith—child interest—means that organization of learning materials cuts across narrow subject-matter compartment lines in favor of broad areas of interest.

> The old school organized knowledge into many minute disparate, academic departments. In the upper grades of some of the new schools the initiating center of organization is the interest of the child in some contemporary institution or problem. In the lower grades the focus is the immediate school scene. In the higher grades the emphasis shifts to adult society; in the foreground always stand the fundamental movements or trends, the crucial institutions or problems of contemporary life.
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> All this does not mean that the new school entirely avoids school subjects but the subjects in these schools differ materially from those of the

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[48] Rugg and Shumaker, *op. cit.*, p. 60.
[49] Rugg and Shumaker, *op. cit.*, p. 61.

> formal school. . . . They are concerned at bottom with big concepts, themes, movements, that explain broad, fundamental phases of human life.[50]

According to Rugg and Shumaker, the new school treats the minimum essentials, on which the old school spent its time and energy and which "are largely skills," as mere by-products of the educative process. Yet, they say, the new school has usually succeeded in teaching these minimum essentials more adequately and in less time than the old school.

A fifth article of faith is the conception implied in the phrase "creative self-expression." Rugg and Shumaker contrast the ideals and practices of the old and the new education:

> The spirit of the old school was centered about social adjustment, adaptation to the existing order. The aim of conventional education was social efficiency. Growth was seen as increasing power to conform, to acquiesce to a schooled discipline; maturity was viewed from the standpoint of successful compliance with social demands.
>
> In the new school, however, it is the creative spirit from within that is encouraged, rather than conformity to a pattern imposed from without. . . . Art in the new school is permitted, in the old it was imposed. . . . But whatever the route, the medium, the materials—each one has some capacity for expression. . . .
>
> The new schools are providing "drawing out" environments in sharp contrast to the "pouring in" environments of the old.[51]

A sixth fundamental concept of Progressive education is involved in the provision of opportunity for development of individual personality and, at the same time, for social adjustment to the point that successful living with the group is possible. Whereas the old school overemphasized competition at the cost of unfitting the child for successful social living, according to the Progressives, the new school

> . . . Encourages the child to be a distinct personality, an individualist, to believe in his ability—but of course, not to an unjustifiable degree. It sets up situations which provide constant practice in cooperative living. . . .

[50] *Ibid.*, pp. 61–62. [51] *Ibid.*, pp. 62–63.

> The new school bridges the gap, therefore, between the development of individuality on the one hand and successful social participation on the other by insisting that the true development of the individual and the fulfillment of personality are best attained as one expresses himself most successfully and adequately with others and toward others.[52]

Such is the creed of the Progressives as sketched by Rugg and Shumaker. Other writers would differ somewhat in statement of principles or in details but a comparison of this list of principles with those listed by Kandel[53] will serve to demonstrate the representative character of the analysis of Rugg and Shumaker. As can be inferred from the statement of its basic principles, the Progressive movement implies not only reorganization of subject matter and, to a great extent, new and different subject-matter content in education, but also the use of new and different methods of instruction.

TEACHING METHODS IN THE PROGRESSIVE PROGRAM

A characteristic feature of Progressive educational procedures is the so-called *project method.* The term *project method,* as Bode points out,[54] has been interpreted in three different ways: (1) as a method of organizing educational materials—a method of curriculum construction; (2) as a method of teaching; and (3) as "wholeheartedly purposeful activity." This last is the definition of Kilpatrick.

The term *project* came into education by way of vocational-agriculture courses in high school; as both Bode[55] and Thayer[56] point out, the primary purpose of the project is to supplement regular school work. Likewise, the project involved application of principles learned in school to life situations and its appeal to the pupil's interest because of its "real life" character was, as Thayer says, "a fortunate by-product rather than an inherent reason for the procedure."

Success of the project idea in agricultural education led to its extension to education in general. The enthusiasm with which pupils carried on project work led teachers to ask, in the words of Thayer, "Why should we not organize the regular course of study around concrete living situations, and thus reduce to a minimum the formal character of school work? Instead of confining projects to the application of a principle or facts previously learned, why not reverse the procedure and organize class work so as to move from the concrete situation to the fact or principle or generalization?"[57] Under this plan, the project came to be more important in the eyes of teacher and pupil than the facts and principles which were to be derived from it.

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[52] *Ibid.,* p. 65.

[53] Kandel, *op. cit.*

[54] Bode, *op. cit.,* pp. 141–142.

[55] *Ibid.*

[56] V. T. Thayer, *The Passing of the Recitation,* Boston, D. C. Heath and Co., 1928, pp. 229–231.

Final extension of the project idea is based on the interest which a real-life problem arouses. Thayer says, "Consequently we find a group of educators defining the project more or less in terms of its purposive character, on the assumption that the real-life character and the applications of knowledge used will take care of themselves once the pupil wholeheartedly identifies himself with the task in hand."[58]

This system involves abolition of traditional subject-matter organization in favor of project-type organization. The most extreme form of this doctrine does not make use of projects fixed in advance but begins with the purposes and suggestions of pupils and develops these as far as they are practicable in stimulating pupil growth. This is Kilpatrick's conception of the project method; "as thus conceived the project method no longer supplements school learning, it constitutes the whole of school learning."[59]

It seems to be a reasonable assumption that "activity movement" refers to the conception of learning as an active process—actually putting into practice the idea of "learning by doing"—and that the project as defined by Kilpatrick represents the procedure to be used. Thus Rugg and Shumaker say of the program of work in the new schools, "According to the new psychology, learning is not a passive, memoriter process; it is a dynamic, assimilative process. The new

[57] *Ibid.*, pp. 236–237.

[58] *Ibid.*, p. 243.

[59] *Ibid.*, p. 246.

schools postulate, therefore, that learning takes place best under those stimulating conditions of real life in which the learner participates in activities which he helps to initiate and for which he sees a need. . . . The vocabulary of the child-centered school resounds with centers of interest, creative activity, pupil enterprises, self-initiated undertakings, open forums and debate, experimentation in shop, kitchen, laboratory, and studio, research in library and afield."[60]

CURRICULUM MAKING IN THE PROGRESSIVE PROGRAM

It will be noted in the foregoing quotations from leaders in the field of Progressive education that considerable care appears to be exercised in characterizing the part pupils are to play in formulating curriculums as "participation." Such a view is clearly opposed to the view, traditional or otherwise, that curriculum making is the sole concern of educational experts, be they teachers, administrators, curriculum specialists, or textbook writers. It is basically opposed to made-in-advance curriculums, and to autocratic domination of pupils by school authorities.

Just how much participation is to be permitted of pupils, however, is not so clear. Rugg and Shumaker specify that the activities in which a learner participates must be those "which he helps to initiate and for which he himself sees a need." Thayer quotes Collings as emphasizing "beginning with the original purposes or suggestions of the pupils."[61] Kilpatrick insists that a learner must take each step in the activity process which includes "purposing" and "planning."[62] Are pupils merely to initiate, and is the teacher to take over from then on? If so, authoritarianism is merely sugar coated; the pupils are fooled into believing that they are participating, whereas they are doing nothing of the kind. Kilpatrick is opposed to such tactics for he deplores using project teaching merely as a means of putting over prearranged subject matter.

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[60] Rugg and Shumaker, *op. cit.*, pp. 68–71.

[61] Thayer, *op. cit.*, p. 244.

[62] Kilpatrick, *Foundations of Method*, pp. 215–216.

Is pupil participation to be 100 percent, with the teacher taking a thoroughly passive role and doing nothing except perhaps restore order should that become necessary? It is hardly conceivable that modern teachers and administrators would take such a view. However, the economic principle of laissez faire as applied to business is an almost exact parallel to this view, and considerable evidence may be gathered to show that not a few teachers, and even whole school systems, actually attempt to conduct classes in strict accordance with this principle. Otherwise, why did Rugg and Shumaker find it necessary to decry the tendency toward "planlessness," and why must such sympathetic critics as Dewey, Thayer, Cobb, and others keep insisting on the point?

A third alternative is to make curriculum building a joint affair, to secure pupil participation through coöperation between teacher and pupils. Mrs. Mossman says, "More important than all these is consideration of what the group will do and how it proposes to begin. What are the pupils to do while living together in this room in the months that are ahead? . . . The individual ideas and proposals of the members of the group may in all sincerity be made vocal if the discussion is honest and the children understand that they really have a participating role in planning the work."[63] Such a plan would indeed seem democratic in the sense of providing equality of opportunity to participate in formulating the course of study. But the exact parts that teacher and pupil are to play in such a program are still not clear. Teachers who participate in the program must have a clear sense of direction; they must have their educational objectives clearly formulated if they are to achieve their aims.

THE PLACE OF MATHEMATICS IN PROGRESSIVE EDUCATION

Perhaps we can discover how the Progressives would solve the foregoing problem by examining what they have to say about the teaching of mathematics. Kilpatrick says:

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[63] Mossman, *op. cit.*, p. 17.

> Arithmetic we shall always need and shall always teach and we shall teach it openly. . . . We learn better . . . when we face a situation calling for the use of the thing to be learned. Other things being equal then, we shall try to teach our arithmetic as it is needed, that is, in connection with situations of actual need. The effect of this will be to find arithmetic in many little pieces scattered along the path of life. These we shall teach as we meet them. As we accumulate in this way a store of arithmetic some of the pupils, particularly the more mathematically inclined, will from time to time put the pieces together and form wholes more or less complete. Later some will specialize in the subject.[64]

To the question of whether there might not be certain minimum essentials which it is necessary to compel a child to learn, Kilpatrick answers:

> I think there are certain things so useful for future progress in school and life, both immediate and more remote, that we should use compulsion if need be to get them, so important that if they are not got otherwise, there would eventually come a time when we should, if need be, drop practically everything else and compel the learning of them. . . . Only I cannot give you a complete list. I think, in fact, it would differ with each child and the conditions surrounding him. My general list for all would certainly be much smaller than you are accustomed to think. Reading would be the main essential, the ability to manage ordinary reading matter. Counting, making change, column addition should be included. These are not all but the list is short.[65]

However, that still leaves opportunity for anyone else to agree with Kilpatrick perfectly, yet extend the list of things he considers essential as far as he likes. Bode asks:

> How extensive is this list of minimum essentials? Kilpatrick says simply that *his* list would be very short. With others the list would no doubt be very long. It is antecedently rather likely that the arguments in favor of the subjects on the short list would apply quite as directly to a longer list. For example, we say that reading and number work are minimum essentials; why should not the same claim be made for history and geography? . . . As long as things are running smoothly the minimum essentials pre-

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[64] Kilpatrick, *Foundations of Method,* p. 357.
[65] *Ibid.,* pp. 365–366.

sent themselves in all sorts of contexts; they are not learned separately. As Kilpatrick says, "we find arithmetic in many little pieces scattered along the path of life. These we shall teach as we need them." But as long as it works out this way there is no need of exercising compulsion. To "drop practically everything else and compel the learning" of the minimum essentials would be an abandonment of the whole idea of starting with a "fruitful experience" and a reversion to the old method of set tasks and prior chosen subject matter. On casual inspection, at any rate, this looks like an oscillation between two extremes. On the one hand the teacher all but fades out of the picture; on the other hand the teacher functions like a drill sergeant in charge of the awkward squad.[66]

Mossman believes that arithmetic was put into the schools originally because home conditions failed to give opportunity for proper development of quantitative concepts. In other words, pupils who learn their arithmetic "in connection with situations of actual need" do not "put the pieces together to form wholes more or less complete," as Kilpatrick believes will happen. In discussing the fact that learning in colonial days was almost entirely "in connection with situations of actual need," Mossman says that "there were only a few situations calling for the abilities to read, write and spell. There were numerous quantitative situations in counting, adding, and measuring, but there was little time in the home for developing in the learners quantitative relationships arising from their own experiences or in developing these relationships into a definite system. Hence a school was established to provide for these deficiencies—a school of reading, writing, arithmetic, and spelling."[67] However, Mossman believes that the old concept of learning these skills by practice—mere repetition—is wasteful of time and stupidly monotonous. She says, "The intelligent effort of the learner to master the technique presupposes his definite purpose to learn. The way, then, to secure such effort is to have the learners engaged in a program of living where adventuring, exploring, and inquiring call for techniques, where skills come to be seen as needed in order to move ahead in the enterprises undertaken. When seen as needed these techniques come to be valued and sought. Thus purposes to learn skills may arise and drill may become truly an adventure for the learner."[68]

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[66] Bode, *op. cit.*, pp. 162–163.

[67] Mossman, *op. cit.*, p. 25.

This utilization of pupil purpose to give necessary mastery of fundamental skills would largely eliminate compulsion in learning Kilpatrick's short list of minimum essentials. Thus, in spelling, "words to be learned become the words that the learner needs in what he is doing but that he has difficulty in spelling. This need for the words as recognized by him, when coupled with pride in carrying what he is doing through successfully, gives him the purpose to learn them."[69]

The learning of number skills may be attacked in the same way, according to Mossman. "Incidental number facts and simple skills arise in the course of daily living and are easy to learn." But the teacher often fails to see the number situations which confront the learner. These number situations of the children, which the teachers need to recognize, may include "the distribution of materials to members of a group, the division of a class into groups, or the approximate cost of the luncheon in a cafeteria as it accrues in choosing dishes for the tray as one goes down the counter." The latter, according to Mossman, is particularly difficult if a learner has only a limited amount of money with which to pay for his meal. That this situation represents a real-life situation, one in which most children will find themselves more and more frequently as they grow older, not even the most captious of critics can effectively deny. However, not every critic will agree with Mrs. Mossman that "as a learner gets individual number facts of this kind, he will come in time to ask how many more such there are and, if he is getting a wholesome, healthy attitude toward achieving in all his enterprises, may readily propose to master all of them. So he may undertake learning all the '7's' after having met some of them and seen them useful."[70]

The curriculum in arithmetic under such conditions will, of necessity, be limited to those phases of the subject which have immediate application. Similarly, Wrinkle would use immediate applicability as the criterion for choosing the mathematics subject matter to be taught in the secondary school. He says, "The only wholly defensible basis on which mathematics may build its claim to a place in the secondary school is in terms of its functional contribution to everyday living. Expressed differently—the outcomes of instruction in mathematics are primarily tool or service skills which are valuable only to the extent that they are usable."[71]

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[68] *Ibid.*, p. 77. [69] *Ibid.*, p. 78. [70] *Ibid.*, p. 81.

THE PROGRESSIVE PROGRAM IN THE SECONDARY SCHOOL

Before taking up the secondary-school curriculum in mathematics it is necessary to say something about the status of the Progressive movement with respect to secondary education as a whole. In general, the Progressive movement has made far less headway in the secondary than in the elementary school. It has made most progress in the primary grades. Rugg and Shumaker ascribe the lack of progress on the secondary-school level to the fact that the high school has served principally as a college-preparatory institution and as such its program has been determined by college-entrance requirements.[72] However, a number of secondary schools have organized their programs along Progressive lines. One of these was the secondary school connected with the Colorado State College of Education at Greeley, of which Dr. Wrinkle was at one time director. Dr. Wrinkle's book is a description of the program he instituted in that school. As has been mentioned previously, there is not complete agreement among Progressives on details but the program proposed by Wrinkle is at least suggestive of what Progressive principles imply and, in the absence of any official program, may be used to indicate the place of mathematics in the curriculum of a Progressive secondary school.[73]

[71] William L. Wrinkle, *The New High School in the Making,* New York, American Book Co., 1938, pp. 180–181.

[72] Rugg and Shumaker, *op. cit.,* pp. 44–45, 90, and 318.

[73] That is, unless the report of the Commission on the Secondary School Curriculum of the Progressive Education Association entitled *Mathematics in General Education* be taken as such. In the author's opinion, Wrinkle's program follows more logically from Progressive philosophy than does the Commission's report.

UTILITY AS CRITERION FOR CHOICE OF SUBJECT MATTER

Wrinkle recognizes the contribution which mathematics has made to civilization but argues that this is not a reason for making the study of mathematics an essential part of general education, since the mathematical contributions which are so important to the progress of civilization have been made by a few gifted individuals. Wrinkle says that there are four values "recognized or claimed" for the study of mathematics—cultural, disciplinary, preparatory, and functional. He considers the cultural values of mathematics "largely mythical" and the disciplinary not demonstrable.[74] The idea of preparatory value is, of course, incompatible with the conception of education as life, as the satisfaction of felt needs, rather than as preparation for life. Hence, to Wrinkle the only defensible educational value of mathematics is its functional value. He says, "From the functional point of view, the program should include such mathematics as may be needed by the individual in the solution of his problems or the satisfaction of his interests. Mathematics is involved in many everyday life situations. An analysis would reveal, however, that for the most part these situations involve simple computations, such as making change, adding up items on a bill, figuring tax, and other activities of similar simplicity."[75]

It is doubtful whether Wrinkle includes in his "life situations" the obligation of an adult to understand, as an intelligent voter, questions of public as well as individual finance. If he did, he would recognize that questions such as amortization of bond issues and the effect of proposed public expenditures on tax rates are not quite as simple mathematically as he implies. Apparently he, as well as some other "child-centered" enthusiasts, ignores the interests of the state in establishing and paying for public education.

Application of the functional-value test to instructional materials, Wrinkle says, would undoubtedly scrap much of the present curriculum but would bring in many new problems not touched by the conventional program.

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[74] Wrinkle, *op. cit.*, pp. 180–181.
[75] *Ibid.*, p. 181.

> This proposal would not mean the elimination of mathematics from the school curriculum. There are many functional skills and understandings. . . . Practically all individuals some day buy or are sold insurance. . . . Taxation is a problem. . . . Consumer buying is an area in which everyone is involved. Daily we are confronted by . . . advertisements encouraging purchases on a six percent deferred payment basis. How many . . . can actually tell what the percent rate of interest really is? New provisions for social security, insurance, and pensions have recently added new problems, some understanding of which is desirable. . . . In the junior high school levels, mathematics teachers have not hesitated to move into the social-commercial-vocational-arts field to secure problems of a more vital nature. Why shouldn't mathematics in the senior high school be concerned with these vital problems of everyday life?[76]

Algebra and geometry will be retained by Wrinkle to meet college-entrance requirements and the few students who wish to take these courses will be given the opportunity, but only in the last years of secondary school. Other than in the college-preparatory courses in algebra and geometry, the program of instruction in mathematics will be incidental in character. Here Wrinkle follows Kilpatrick in that in certain situations he abandons individualistic methods in favor of traditional treatment of mathematics. This in itself indicates that the revision of the educational program which he proposes is not basic—it does not imply a thoroughgoing reconstruction of traditional teaching procedures. Even Wrinkle does not depend on the incidental method when specific standards of achievement must be met, as in the case of college-entrance requirements.

MATHEMATICS TO BE TAUGHT INCIDENTALLY

Wrinkle apparently makes no provision for students who are not preparing for college to study the theoretical phases of mathematics. He suggests that the actual program in secondary-school mathematics will depend upon the success achieved by elementary schools in developing a reasonable command of the fundamentals of mathematics. He says:

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[76] *Ibid.*, p. 182.

> Assuming that the student moving into the secondary school has a reasonable command of the fundamental abilities, there should be no formal courses in mathematics until the college preparatory algebra level would be reached.
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> Mathematics instruction should be incorporated into the curriculum wherever it is needed. The mathematics teacher instead of having a class reporting to her daily to study problems in mathematics would, in co-operation with the other teachers, teach the mathematics involved in the other areas of experience at such time as the mathematical abilities would need to be taught. Specifically, instead of teaching the equation in ninth grade algebra the mathematics teacher would work with the physics or chemistry teacher, if that is where the equation was first used, and would there teach the equation in the situation where its use is needed. Likewise, the mathematics teacher would work in co-operation with the social studies teacher in the study of insurance, taxation, budget making, installment buying, and the other problems of social significance.[77]

To maintain the fundamental skills which the individual needs in normal living, a program of testing periodically to see if students have maintained a reasonable facility and level of accuracy in these skills is suggested.

> To provide this exercise, it was first necessary to determine what the fundamental skills were which should be maintained. Among the simple skills which might be identified would be the skill in division of whole numbers by such common fractions as one-half, one-third, one-fourth. Simple test exercises were prepared for the purpose of testing the student's possession of this skill. The tests were brief, requiring five minutes or less for administration. Immediately following the completion of the test the students, with the direction of the teachers, would score their own papers. Students who had experienced no difficulty in completing a reasonable number of the problems accurately had demonstrated that they had command with respect to that particular skill. If the student had lost command of the skill, his deficiency would be reflected in his test record. He would then be a candidate for reteaching.
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> This testing-teaching procedure might take place in any class which the mathematics teacher might wish to utilize. It might be practiced on

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[77] *Ibid.*, p. 183.

any level of the school organization, beginning with the lowest grade of the secondary school and extending on up as far as there would be students who would not have command of the skills. . . . In this manner provision is made for testing, reteaching, retesting continually throughout the secondary school program on all levels until the ability to maintain the skill had been unquestionably established. Only at that point would the practice be discontinued.[78]

This last suggestion obviously implies the "stamping in" of desired responses in the best bond-psychology tradition. Kilpatrick implies the same sort of procedure in his short list of minimum essentials to be learned by compulsion if necessary. At least, nothing is said or implied to deny the desirability of such procedure.

Wrinkle mentioned that at the time his book was written the program he suggests had been in actual operation for four years but had applied to too few cases and to too selective a group to give conclusive evidence of its adequacy. He insisted, however, that "no observable damage" to the students had resulted.

SUMMARY AND APPRAISAL

The Progressive program may be summarized, in so far as it applies to mathematics, by saying that its characteristic method is the project method (it depends upon incidental or instrumental learning) and uses two different criteria for choice of subject matter—immediate interests of pupils and immediate utility. An entirely different program—a program based on the theoretical aspect of mathematics—is provided for the students who expect to enter college. Apparently, the mathematics program in the secondary school is to be dualistic in nature with one phase based on pupil interests and utilitarian values and the other on traditional treatment of the theoretical aspects of mathematics, to be imposed upon college-preparatory students in the best authoritarian manner.

The first phase—the Progressive part of the program—is formulated upon the assumption that the theoretical phase of mathematics has no educative significance for most students. In other words, the revolt of the "rebels and protestants against the regimentation of the formal school" amounts, for the non-college-preparatory course, merely to taking the side of the practical in the age-old quarrel as to whether we should stress the practical or the theoretical.

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[78] *Ibid.*, pp. 185–186.

So far as the typical teaching method of the Progressives—the project method—is concerned, the comments of Bode are pertinent:

> In a real project the connection between studying and carrying on the job must be sufficiently close so that the job determines directly what is to be studied and gives opportunity to try out the new knowledge from time to time. . . . When this relationship is maintained the studying or learning is clearly instrumental to a purpose, and this fact puts us on track of a definition. A boy working on a project like the raising of corn would organize his intellectual work with reference to this particular topic. . . . His purpose would not be to acquire knowledge but to grow corn. Learning would be simply a means to an end and not an end in itself. The project method . . . is the method of instrumental or incidental learning.[79]

Bode suggests that there is a real place for the project method in education but that it cannot be made to cover the whole field. He says:

> By definition it (the project method) takes no account of either logical organization or "social insight." Its spirit is the spirit of immediate practicality which is the spirit of an exclusive vocationalism. This is no objection to the method unless we apply it too widely. If we do so, we find that our practicality overreaches itself. Learning that is limited to this method is too discontinuous, too random and haphazard, too immediate in its function unless we supplement it with something else. Perhaps children may learn a great deal about numbers from running a play store or bank, but this alone does not give them the insight into mathematics they need to have. . . . Learning for immediate purpose, or incidental learning, is too much a hit-or-miss affair—it dips in here and there, but it gives no satisfactory perspective, no firm hold on fundamental principles.[80]

Bode comments that this is not a castigation of the project method as such but an attempt to show its limitations. Kilpatrick tries to redefine the project method so as to eliminate its deficiencies. He defines it, as we have seen, as "wholeheartedly purposeful activity," a definition in terms of the attitude of the learner toward his work. The importance of Kilpatrick's conception of the project method, says Bode, does not lie in its emphasis on purposeful activity but in its bearing on curriculum construction and teaching method.[81] He says further that, although the doctrine of purposeful activity at first seems quite revolutionary, later it appears that activity should be purposeful only as far as it can be purposeful. It is quite likely that no one would dispute this latter idea.

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[79] Bode, *op. cit.*, pp. 148–149.

[80] *Ibid.*, pp. 150–151.

As to the emphasis on initiative and purposive activity, Bode quotes Dewey's remark that thinking requires suggestions, there being no spontaneous generation in the mental life. The suggestions must come from somewhere and who is better qualified to suggest than a teacher? "Education means guidance and direction which it is the function of the teacher to supply," says Bode.[82] He comments further that Kilpatrick's suggestion, previously quoted, that "some of the pupils, particularly the more mathematically inclined, will from time to time, put the pieces together and form wholes more or less complete and that later some will specialize in the subject," is not carrying out a constructive program but merely harking back to Rousseau.[83]

Universality of application of the method favored by the Progressives is seriously questioned, even by some Progressives. Stanwood Cobb, founder and early president of the Progressive Education Association, lists as "limitations of activity education" the following: (1) the assuring of definite cultural results is neglected; (2) the tools and techniques of learning cannot be learned by the project method but can only be motivated by it; (3) lazy, unambitious and slow children respond better under academic pressure and discipline; (4) retarded children above borderline intelligence need thorough drilling in the techniques of reading and arithmetic more than they need activities.[84]

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[81] *Ibid.*, p. 160. [82] *Ibid.*, p. 164. [83] *Ibid.*, pp. 164–165.

[84] Stanwood Cobb, *New Horizons for the Child*, Washington, Avalon Press, 1934, p. 115.

The contention that mathematics owes its place in the curriculum to college-entrance requirements is not supported by historical evidence. As was pointed out in a previous chapter, mathematics came into the academy curriculum in response to the demands of students who were not preparing for college entrance. Algebra, geometry, and even arithmetic were originally college subjects and were added to the list of entrance requirements only after training in these subjects became so general among entering students that college classes in the elementary phases of mathematics were no longer needed.

The Progressive program seems to ignore the desirability of enabling pupils to understand life situations of any scope larger than their own individual needs and experiences. It fails to give pupils an adequate understanding of the needs of others or to fit them for participation in the public affairs necessary for the public good in a democracy.

Hence, it seems that the program of the Progressives is inadequate in so far as mathematics is concerned. It is based upon a method of instruction—the project method—that will not admit of universal application. An adequate grasp of mathematical principles cannot be acquired incidentally, as the Progressives advocate. They present a program based on the assumption that only the practical aspects of mathematics have value for everybody and that the theoretical aspect is of value only to those who expect to enter college. The program provided for college-preparatory students is, apparently, the traditional program. In the program designed for the great majority of students—those who do not expect to enter college—the Progressives neglect the theoretical phase of mathematics entirely and attempt to justify that neglect by appeals to theories of psychology and learning that are themselves open to serious question. In blaming the colleges for bringing mathematics into the secondary-school curriculum they mistake the *results* of trends for the *causes* of those trends. Consequently, it is not surprising that a program based on such confused and contradictory conceptions of the educational process should prove to be wholly inadequate from whatever viewpoint—student's, teacher's, employer's—it may be examined.

Apparently, even the Progressives themselves came to realize some of the shortcomings of their program for mathematics education. In 1940 a committee of the Commission on the Secondary School Curriculum of the Progressive Education Association published a report entitled *Mathematics in General Education* presenting the results of an examination of "the study and teaching of mathematics for their values in relation to the whole process of general education."[85] The committee which was responsible for this report was decidedly not in complete accord with Wrinkle and other prominent Progressives who insist that the study of mathematics has only utilitarian values. On the contrary, the committee believes that "The teacher of mathematics can make certain indispensable and distinctive contributions to the attainment of the broad purposes of general education."[86] In fact, it even goes so far as to say, "The Committee shares the conviction that, divested of much of its conventional content and formal organization, mathematics as a mode of thought and an instrument of analysis has an indispensable function in general education."[87] However, the cautious qualification "divested of much of its conventional content and formal organization" robs this commitment of much of its force and leaves one still wondering just what mathematics, and how much, has, in the opinion of the committee, "an indispensable function in general education."

We find that, according to the committee, the study of mathematics may increase the social sensitivity of a student;[88] it may contribute to his growth in "sensitivity to the esthetic quality of experience";[89] and "a taste for strict logical deduction, a hearty respect for the power of reasoning, a confident faith in the value of sound inference, all these aspects of an appreciation for logical thinking are particularly appropriate objectives of mathematical study."[90] Also, according to the committee, the study of mathematics may have all the values mentioned above in addition to helping meet the needs of students when "quantitative data and relationships or the facts and relationships of space and form are encountered." Obviously, therefore, and in distinct contrast to Wrinkle, the committee was of the opinion that the study of mathematics has cultural values distinct from its utilitarian applications. However, despite this somewhat belated recognition on the part of the Progressives that study of mathematics at the secondary level is educationally respectable—even desirable under certain conditions—it must be remembered that Wrinkle's proposals were more in keeping with fundamental Progressive philosophy and practice than were the views of the committee.

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[85] *Mathematics in General Education,* a report of the Committee on the Function of Mathematics in General Education for the Commission on the Secondary School Curriculum of the Progressive Education Association, New York, Appleton-Century-Crofts, Inc., 1940, p. vi.

[86] *Ibid.,* p. 43.

[87] *Ibid.,* pp. 11–12.

[88] *Ibid.,* p. 48.

[89] *Ibid.,* p. 49.

CORE CURRICULUMS

Specific-objectives proposals for educational reform were principally aimed at modification of content and improvement of teaching techniques within the framework of the existing curricular organization. The subject-matter units proposed by Morrison and his followers did not, in general, cut across traditional subject-matter boundaries. The Progressive program with its emphasis on child interest, creative self-expression, and pupil-purposed activity in real-life situations obviously calls for a new type of curricular organization—one in which the barriers between various subject-matter areas and between subjects within an area would be largely eliminated. Preliminary efforts in this direction took the form of attempts to combine the content of two or more traditional courses into a single "fused" or "integrated" course. The general mathematics course which has come to be the typical ninth-grade mathematics course is an example of an integrated or fused course. Attempts to organize courses in which emphasis is placed on the elements common to closely related fields such as mathematics and science have not been generally successful.

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[90] *Ibid.*, p. 50.

However, any curricular organization in terms of subject-matter areas would not be in keeping with the spirit of Progressive education with its emphasis on pupil needs and pupil interests. Furthermore, with increasing democratization of the secondary-school student body had come changes in pupil needs and pupil interests. Williams summarizes these changes as follows:

> When the high school student body was made up of youth who came almost exclusively from homes in which the fathers were engaged in professional and managerial occupations, as shown by Counts in his famous study of 1922, the students were youth who had come up through a childhood where almost unconsciously they had absorbed in their everyday life correct speech habits, social amenities and courtesies, good taste in literature, music, and art, attitudes contributory to social habits, and other elements. The needs of these youth were for formal schooling, for knowledge which comes from subject matter found in school textbooks, for training which would fit them to go forward into and through higher institutions of learning. To this group, however, there has been added, in an ever increasing proportion over a twenty-year period, representatives of homes whose parents are engaged in other forms of occupation. Many of these youth have not spent their childhood in an environment conducive to social living, aesthetic interests, interests in wholesome recreation, pride in correct language usage, and scores of other desirable ways of life. These youth need subject matter knowledge and, still more, opportunities to engage in enterprises in which constructive life experiences can be obtained. In addition to learning what and how to study, they need to learn how to live richly and abundantly.[91]

An obvious solution to the problem presented by these changes in pupil needs and pupil interests is to set up a core curriculum based on the needs of *all* pupils under such categories as "personal living," "immediate personal-social relationships," "social-civic relationships," and "economic relationships." According to Giles, McCutcheon, and Zechiel, this core curriculum providing a common body of growth experiences for all students would be organized into units of study designed to meet these needs "in such a way as to develop the *characteristics of personality* needed for effective functioning in a democratic society."[92] They continue:

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[91] L. A. Williams, *Secondary Schools for American Youth*, New York, American Book Co., 1944, p. 241.

> Believing that the *characteristics of personality* needed for effective participation in a democratic society can be developed only through the practice of democratic procedures, the staffs developing such core courses make certain that these are characterized by wide participation of the students in planning both the scope and the sequence of the curriculum. The activities of a class follow no fixed or internally logical sequence, such as would be defined by a textbook or a course of study. This does not mean that the activities have no logical organization, but that the procedure of problem solving and an application of intelligence form the basis on which logic is achieved.[93]

Obviously, the amount of sequential work in a subject such as mathematics in a core curriculum so conceived and so developed will be greatly restricted. In fact, only utilitarian, quantitative concepts of mathematics would probably find place in such a curriculum. The curricular organization proposed by Wrinkle with all formal study of mathematics restricted to students definitely preparing for college entrance exemplifies exactly the scope and spirit of the mathematics instruction implied by the core-curriculum concept. What has been said concerning the strengths and weaknesses of Wrinkle's proposals applies with equal force to proposals for developing core curriculums in so far as instruction in secondary-school mathematics is concerned.

There is a real and pressing obligation on the part of secondary schools to meet the personal and social needs of that great body of students whose social and experiential background has been unduly narrow and meager. But it is only a partial solution to the problem to concentrate on experiences designed "to develop the characteristics of personality" and to neglect entirely pupil needs in other areas such as knowledge of subject-matter. Today's students need both an opportunity for personal development and opportunity to gain the organized sequential information found in textbooks. They need, as Williams says, to learn "what and how to study" in addition to learning how "to live richly and abundantly."

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[92] H. H. Giles, S. P. McCutcheon, and A. N. Zechiel, *Exploring the Curriculum*, New York, Harper & Brothers, 1942, p. 44.

[93] *Ibid.*, p. 45.

CONCLUSIONS

Thus we see that the Progressives arrived at essentially the same conclusions regarding secondary mathematics as did the advocates of the specific-objectives approach to curriculum making and education. Both groups assumed that the educative value of mathematics is derived entirely from its occupational utility. Both assumed that transfer of training took place only by means of identical elements present in both situations and, therefore, that the cultural values to be derived from the study of mathematics were comparatively insignificant. The identical-elements theory of transfer accepted by both groups has been largely discredited, and the narrow, utilitarian conception of mathematics and its educational values basic to the programs advocated by these groups entirely neglects the educational values to be derived from the study of theoretical mathematics. Hence, neither group is able to solve the fundamental problem presented by the conflict over the place and function of mathematics in secondary education. Both groups, instead of seeking ways and means of resolving the points at issue in this conflict, entered into the controversy on the side of the advocates of the practical. The doctrine of specific objectives and the Progressive movement have served, therefore, to increase rather than diminish the confusion regarding the place and function of mathematics in secondary education.

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Chapter 5

The Place of Mathematics in Secondary Education

WHAT ARE THE ASSUMPTIONS BASIC TO ANY PROGRAM OF EDUCATION?

The conflict of opinion over the place and function of mathematics in secondary education has its origin in conflicting ideas about the nature of the learning process, the function of subject matter in education, and even the aims of education itself. It is readily evident that any proposal for resolving these differences will involve acceptance of certain basic assumptions relative to the nature of the society in which the educational program is to function, the nature of the learning process, methods of teaching, the method of selection and the function of subject matter. For example, teaching method refers to the procedures by which a teacher utilizes subject matter to effect in his students those changes in attitude, in informational background, in technical skills, and in habits of thought which constitute the accepted goals of education. In any society, the habits of thinking, skills, subject-matter content, and attitudes most favored as educational objectives will be those most in keeping with the existing social pattern. The favored teaching methods will be those deemed most effective in utilizing chosen subject matter to achieve these objectives.

ASSUMPTIONS AS TO NATURE OF SOCIAL ORGANIZATION

The starting point in any analysis of teaching method or theory is, then, an analysis of the social organization and the characteristic habits of thought, attitudes, and beliefs which exemplify that society. Dewey says:

> Since education is a social process, and there are many kinds of societies, a criterion for educational criticism and construction implies a particular social ideal. The two points selected by which to measure the worth of a form of social life are the extent in which the interests of a group are shared by all its members, and the fullness and freedom with which it interacts with other groups. An undesirable society, in other words, is one which internally and externally sets up barriers to free intercourse and communication of experience. A society which makes provision for participation in its good of all members on equal terms and which secures flexible readjustment of its institutions through interaction in the different forms of associated life is in so far democratic. Such a society must have a type of education which gives individuals a personal interest in social relationships and control, and the habits of mind which secure social changes without introducing disorder.[1]

The type of social organization characterized by Dewey as *democratic* is the type of social organization most in keeping with the American tradition and with the ideals of the American people.

A democracy is, according to Dewey, "more than a form of government; it is primarily a mode of associated living."[2] Bayles defines democracy as "*a form of socio-governmental organization in which there is equality of freedom or opportunity to participate in making decisions on matters of group or individual concern, and equality of obligation or responsibility to abide by such decisions and carry them out.*" (Italics in the original.)[3] For the individual, to characterize the society of which he is a part as democratic is to indicate that he has both freedom of choice and action and responsibility for the consequences of his exercise of that freedom equal to that of any other individual in the group. For the group, according to Bayles, "Democracy is a way of making choices. It does not represent a particular set of choices, except for choice of the generalized method."[4] He describes the democratic method as "one of cooperation, of mutual sharing both of obligations and benefits. Democracy is a certain kind of cooperation, characterized by the absence of special privilege and of barriers which would stand in the way of equal opportunity on the part of all members to participate or not participate in common interests."[5] Furthermore, the phrase *equality of opportunity* should not be interpreted to mean identity or sameness of opportunity for all. Bayles says:

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[1] John Dewey, *Democracy and Education,* New York, The Macmillan Company, 1920, p. 115.

[2] *Ibid.,* p. 101.

[3] Ernest E. Bayles, *The Theory and Practice of Teaching,* New York, Harper & Brothers, 1950, p. 33.

> Equality of opportunity refers as much to differences among persons as to likenesses. Democracy means equality of opportunity to be different from others as well as to be like them. Cooperation means bringing together diverse talents to work for common ends, and democracy means equality of opportunity to participate in determining the kind and degree of diversification which shall be practiced. To say that democracy means any particular feature of the American Way other than the generalized method of participation in determining what the various features of that way shall be, is to stretch the term beyond its reasonable meaning.[6]

To characterize a society as democratic is, therefore, to describe the method of reaching decisions favored by members of that society and to define their responsibilities for and their attitudes toward decisions so taken.

However, to reach decisions by democratic means does not thereby insure the wisdom of those decisions. The wisdom or unwisdom of decisions taken by democratic means is dependent upon the competence and integrity of the individuals participating in the decisions. For a democratic society to prosper, its members must be wise—must possess the competence and integrity needed to make wise decisions. The function of education in a democratic society is to enable the people to become wise; it is to develop in each individual competence to exercise wisely the freedom of choice and action that is his and willingness to assume the responsibilities consequent upon that freedom.

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[4] *Ibid.*, p. 29.

[5] *Ibid.*

[6] *Ibid.*

THE AIM OF DEMOCRATIC EDUCATION

The world of today is characterized by rapid economic and social change. Citizens of a democratic society in a changing world must make crucial decisions in novel situations. Therefore, the educational program of such a society "should promote on the part of every individual and in the highest possible degree the ability to wrestle with problems independently—to think reflectively."[7] As Bayles says, "The keystone of the arch of democratic educational objectives is the ability to solve problems capably and independently, as they arise."[8]

To be able to solve problems as they arise, an individual must be able to think reflectively; he must also possess, or know how and where to obtain, pertinent information about a problem situation sufficient to provide necessary insight into the significant factors in that situation. But, in a changing world, information and ideas that are pertinent to a situation and adequate to control it today may prove inadequate tomorrow. Therefore, an individual in a democratic society facing such a changing world must be continually revising his information about his world. His insights into the factors and relationships of his situation which have meaning for him in the light of his purposes—his outlook on life—must be in continual change and reconstruction, looking toward improvement of that outlook. Moreover, problems do not arise already labeled; a significant phase of the competence of a competent thinker is to be able to recognize that which requires thought and that which does not. A program of democratic education, then, must develop individual competence not only to solve problems but also to identify problems.

[7] *Ibid.*, p. 34.

[8] *Ibid.*, p. 35.

We assume that improvement of an individual's outlook on life implies (1) increase in the number and scope of factors covered by that outlook—*enhancement*—and (2) increase in the degree to which such factors are brought into agreement with one another—*harmonization.* Such an assumption means that areas in which improvement is required, and therefore areas which involve problems, are those representing (1) inadequate coverage of pertinent or influential factors—*inadequacies*—and (2) factors which are, or at least appear to be, out of keeping with one another—*conflicts.* The implication of this line of thought is that, as a covering-end for education, we should seek *"to stimulate a student toward becoming an independent learner while at the same time he is developing an enhanced and more harmonious outlook on life."*[9]

THE SUBJECT MATTER AND METHODS OF DEMOCRATIC EDUCATION

The subject matter of a program of democratic education includes the problems and experiences of life itself and, in particular, those aspects of life which appear to be inadequate or inconsistent—the phases of life which require enhancement and harmonization. One function of subject matter is, therefore, to provide an individual with broadened knowledge of his world—to increase his stock of ideas. This is termed *enhancement* of his outlook on life. Outlook, as used here, means essentially the same as Dewey means by experience when he says that education is "that reconstruction and reorganization of experience which adds to the meaning of experience and which increases ability to direct the course of future experience."[10] We may say that the progressive reconstruction of experience is a process of reflection upon a poor outlook, looking toward a better one. An individual's outlook becomes better as it is broadened in scope—as its coverage is increased—and as it becomes more harmonious. His outlook becomes more harmonious through a process of relating ideas to ideas. As a pupil gets new ideas, he must learn to examine them critically to determine whether they agree with available data and with other ideas considered acceptable by him. The process of promoting agreement among ideas and between data and ideas is termed *harmonization.*

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[9] Ernest E. Bayles, "A Philosophy for Science Teaching," *School Science and Mathematics,* 39:805–811 (December, 1939).

[10] Dewey, *op. cit.,* p. 89.

Bayles suggests that enhancement and harmonization may be used as criteria for the choice of subject matter. He says, "Therefore, in order that the work of a classroom may assist a pupil to develop 'a more competent knowledge of and ability to cope with his world,' each unit of study must represent a satisfactory answer to the questions (a) 'Does it contribute toward enhancement of outlook?' and (b) 'Does it contribute toward harmonization of outlook?' "[11]

In a program of democratic education, then, the acquisition of knowledge—gaining mastery of subject matter—is considered only as a means to the end of progressive reconstruction and reorganization of experience. Improvement of outlook is through reflection and reflection is also the process by which an individual becomes a more independent learner.

Traditionally, the school centered its attention on subject matter and assumed that thought about that subject matter by a learner could be taken for granted. More recently, there has been a tendency to focus attention exclusively, or almost exclusively, on method and to assume again that reflection will follow automatically. As Dewey says, however:

> While we may speak, without error, of the method of thought, the important thing is that *thinking is the method of an educative experience. The essentials of method are therefore identical with the essentials of reflection.* They are first that the pupil have a genuine situation of experience—that there be a continuous activity in which he is interested for its own sake; secondly, that a genuine problem develop within this situation as a stimulus to thought; third, that he possess the information and make the observations needed to deal with it; fourth, that suggested solutions

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[11] Ernest E. Bayles, "The Relativity Principle as Applied to Teaching," *University of Kansas Bulletin of Education,* 4:7 (February, 1940).

occur to him which he shall be responsible for developing in an orderly way; fifth, that he have opportunity and occasion to test his ideas by application, to make their meaning clear and to discover for himself their validity. [Italics mine.][12]

It is particularly pertinent to note that Dewey does not say that we learn by doing. He says that "thinking is the method of an educative experience." Doing may and usually should come into the learning process but, when an individual can learn without doing, time and effort are saved if the doing is omitted. Books are of value in education because they permit learning without doing. It is true that a problem may be recognized as the result of activity and that further activity may be required in solving it, but the educative value of any experience is derived from the insight it gives into an improved way of life and not from the activity just as an activity.

Moreover, studies of the learning process seem to indicate that learning is not a gradual process but proceeds by sudden jumps or spurts. It appears that a given stimulus does not always produce the same reaction in an individual. On the contrary, the response made to a given stimulus is conditioned by (1) what the reacting individual wants to achieve (his goal), (2) the total situation in which he finds himself (his environment), and (3) the knowledge which he possesses as to the most desirable way of achieving his goal in view of the situation in which he finds himself (his insight into his situation). Insight may be defined as recognition and understanding of the significant factors in a situation. An insight may involve, for example, comprehension of the relationships existing between a goal and its method of achievement. Learning, according to the *goal-insight* theory, consists in discovery of progressively easier or more pleasing paths to a desired goal. As Bayles puts it, "As to learning, it is evident that, with a given goal and a given environment, action is dependent on insight. Altered insight means altered comprehension of the way in which a goal is to be achieved and, therefore, an altered sequence of acts in achieving it. Learning thus becomes a process of *coming to sense a new set of relationships—developing new insight.*" (Italics in the original.)[13]

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[12] Dewey, *op. cit.*, p. 192.

Insights may be generalized. That is, a learner may come to recognize that relationships governing a particular situation are typical of other situations as well and that insights gained in solving one problem afford clues to the solution of related problems. This carry-over of learnings from one situation to another is called *transfer of training*. It is through transfer of training that what is learned in school becomes useful and usable out of school. Evidently, therefore, school programs should emphasize learnings which have high transfer value. Transfer of training takes place, according to Bayles, whenever a learner recognizes that ideas and principles developed in one situation are applicable in another and that his present purposes will be served by making the transfer.[14]

> If in-school training is to transfer in the greatest possible degree to out-of-school behavior, it behooves us to make students acutely aware of the possibilities of transfer. We must bring them, as frequently as possible, to see the relationships between in-school and out-of-school situations. In having such relationships repeatedly brought to their attention, the students will become progressively more sensitive to the existence of common or universal features in behavior, more aware of the possibilities of discovering and using such relationships, more likely to discover and apply them for themselves, and thus more likely to transfer previous training to new areas of experience.[15]

General ideas or principles are readily applicable in diverse situations, hence have high transfer value. Whenever there is conscious application of general theory to a specific situation, transfer of training takes place. But there will be no conscious application of general theory to specific situation—no transfer of training—unless that situation poses a problem which the individual wants to solve. For, as Bayles suggests, transfer of training does not take place, even though an individual is aware that opportunity for transfer exists, unless his purposes at the time make it seem desirable. Typically, therefore, transfer of training will occur as a consequence of reflective thinking about a problem situation, although the sensing of similarities or relationships which leads to the discovery of opportunities for transfer of training need not, as Bayles points out, always be on the level of full and clear consciousness.[16] However, "The more clearly or fully the relationships are sensed or comprehended, the greater is the likelihood of transfer."[17]

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[13] Ernest E. Bayles, "Drill, or Thrill, in Education?" *Elementary School Journal*, 40:30 (September, 1939).

[14] Bayles, *The Theory and Practice of Teaching*, p. 97.

[15] *Ibid.*

It is evident that reflective thinking—the active canvassing of one's store of experiences seeking a clue to the solution of a problem—is a means of discovering opportunities for transfer of training. Reflection is thus seen to be a means by which previous experiences may be utilized to give control of future experiences. Thus reflection-level thinking is, as Dewey says, the method of an educative experience.

A program of democratic education can be summarized as follows: Learning is a process of gaining new insights. Reflective thinking is the method of an educative experience. The subject matter of education is the inadequacies and disharmonies in present outlooks on life. An individual is competent to cope with life to the degree that his knowledge of his world is adequate, pertinent, and harmonious. Therefore, an educational program can aid a citizen of a democratic society to achieve competence to cope with the world only if it stimulates him to seek constantly to improve his outlook on life, if it helps him to become an independent learner, one who is able to recognize inadequacies and inconsistencies in his environment and in his attitudes toward that environment and who knows how to attack the problem of removing the inadequacies and resolving the conflicts so discovered. The aim and function of an educational program for a democratic society is to develop in the individual the power to learn independently and to think reflectively about what he learns and about the problems which confront him as he attempts to control his environment—"to direct the course of subsequent experience."

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[16] *Ibid.* [17] *Ibid.*

MATHEMATICS IN DEMOCRATIC EDUCATION

Any educational program, in the final analysis, is an attempt to answer the questions, "*What* subject matter shall be taught?" "*Why* should we teach this subject matter?" and "*How* should this subject matter be taught?" Traditionally, the subject matter of mathematics in the elementary school has been arithmetic; the secondary school has offered algebra and plane geometry, sometimes supplemented by other courses such as solid geometry, advanced algebra, trigonometry, commercial or vocational arithmetic, and general mathematics. Now all of these subjects, with the possible exception of general mathematics, are characterized by a relatively rigid, sequential, logically organized body of subject matter. Yet we have indicated that democracy is a way of making choices. An educational program suited to a democratic society should, it would seem, be one in which students learn to make wise choices. Do these rigid, logical, sequential courses of predetermined blocks of subject matter afford students any opportunity to gain experience in making wise choices? Do they afford students any opportunity to gain insights that will be of value to them as citizens of a democratic society? We remember that, in a democracy, each individual is free to choose for himself. Wise choices can be made only if those participating in the decision are informed, if they know clearly what issues are involved, what the alternatives are, and if they select those alternatives which promise the greatest good to the greatest number. That is to say, competence as a citizen in a democratic society implies (1) knowledge of issues in conflict, (2) the ability and disposition to think reflectively about these crucial problems, to select from among possible courses of action those which will prove wisest in the long run, and, most important of all, (3) sufficient confidence in the wisdom and importance of his decision to take the action which his conclusions imply. Does the study of mathematics develop competence in these aspects of life? Does one, through study of mathematics, become better able "to direct the course of subsequent experience"? If so, is this growth in ability to learn independently and to think reflectively a consequence of the subject matter studied, or may it be due to something else—perhaps to the teaching method used?

It has long been general practice to list a number of specific values attaching to a study and to assume that these values will be achieved as the inevitable consequence of pursuing that study. Dewey says:

> As a matter of fact, such schemes of values of studies are but unconscious justifications of the curriculum with which one is familiar. One accepts, for the most part, the studies of the existing course and then assigns values to them as a sufficient reason for their being taught. Mathematics is said to have, for example, disciplinary value in habituating the pupil to accuracy of statement and closeness of reasoning; it has utilitarian value in giving command of the arts of calculation involved in trade and the arts; culture value in its enlargement of the imagination in dealing with the most general relations of things; even religious value in its concept of the infinite and allied ideas. But clearly mathematics does not accomplish such results because it is endowed with miraculous potencies called values; *it has those values if and when it accomplishes these results, and not otherwise*. The statements may help a teacher to a larger vision of the possible results to be effected by instruction in mathematical topics. [Italics mine.][18]

But, as Dewey implies, the educational values of a subject such as mathematics represent, not values inherent in the subject itself, but possibilities to be realized through effective teaching of the subject. Thus he says, "Numbers are not objects of study just because they are numbers already constituting a branch of learning called mathematics, but because they represent qualities and relations of the world in which our action goes on, because they are factors upon which the accomplishment of our purposes depends."[19]

In other words, the study of mathematics has a place in a program of democratic education if, and only if, a student becomes, through study of mathematics, more competent to cope with life.

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[18] Dewey, *op. cit.*, pp. 287–288.

[19] *Ibid.*, p. 158.

Dewey makes no attempt to distinguish between educational values derived from the study of practical mathematics and those gained from theoretical mathematics. As we have seen, he denies that cultural, disciplinary, or practical values are qualities inherent in the subject matter and regards these values as goals for the teaching of mathematics. He considers discipline, for example, to be training in generalized techniques of problem solving. It includes the old idea of discipline (volitional maturity sufficient to force oneself to do disagreeable but necessary tasks) in the sense that one develops an insight into the fact that, even though a task is difficult or distasteful, it can be accomplished and the end to be gained from doing it is worth while. The study of mathematics may, therefore, have disciplinary values in this sense.

But before such disciplinary values can be realized from the study of mathematics, a learner must come to realize that mathematics is worth while—that the techniques and concepts learned in one situation will transfer to other situations. That is, mathematics must be taught in a way which will emphasize the possibilities of transfer of training. Furthermore, as we have seen, reflection is the process by which opportunities for application of theory to concrete situations —opportunities for transfer of training—are discovered. This concept of Dewey's implies, therefore, not a radical change in the subject-matter content of mathematical education, but broadening of content and development of teaching methods which will be effective in stimulating reflection-level thinking.

WHAT SHOULD BE THE AIMS AND OBJECTIVES OF SECONDARY MATHEMATICS?

It is generally agreed that all individuals need some knowledge of the "fundamentals of mathematics." But just which of the techniques and principles of mathematics may be considered fundamental and just what constitutes adequate knowledge of them are disputed questions.

A great many non-mathematicians are of the opinion that mathematics, especially arithmetic, is a "tool subject." Consequently, they believe that it is only necessary for an individual to master such elementary principles and techniques as will enable him to perform the computations he finds necessary in the course of his daily living. Others, and trained mathematicians in particular, insist that the cultural aspects of mathematics have greater educational significance than its utilitarian applications. A satisfactory means of resolving this confusion and conflict of opinion over the place of mathematics in education will not come out of superficial reshuffling of such secondary factors as subject-matter sequences, grade placement, or classroom techniques. The fundamental causes of the dispute are to be found in conflicting philosophies of life, of education, and of mathematics.

In general, the conflict concerns the relative worth, educationally speaking, of the practical—the immediately useful—and the theoretical or cultural aspects of mathematics. But, although presumably it is possible to distinguish between the practical and the theoretical phases of the subject, actually these two phases are not independent entities. They are coördinate parts of a unified whole. Therefore, any program of education which gives major emphasis to one phase and neglects the other tends, to that extent, to skew the subject out of what may be called its natural form.

There have been periods in the history of mathematics when major emphasis was placed on the theoretical aspects of the subject and other periods when practical applications received greater stress. In general, the former have been periods of rapid development in mathematics whereas the latter have been, to a large extent, periods of intellectual sterility and social stagnation. On the other hand, if theoretical advances are to benefit mankind, they must find applications in practical situations. Hence, as the evidence of history seems to indicate, teaching programs which emphasize one aspect of mathematics and neglect the other have, in general, proved incapable of meeting the needs of life.

But how can the study of mathematics contribute to the development of an individual who is competent to assume the duties, privileges, and responsibilities of citizenship in a democratic society? The subject matter of a program of democratic education should be material which will aid one to attain increased understanding of and mastery over his environment. Understanding of and increased degree of mastery over one's environment will come as a consequence of increased knowledge of that environment and from reflection upon the relationships that exist therein. Therefore, the function of subject matter in democratic education is (1) to impart information about the environment and (2) to stimulate reflection upon the problems presented by that environment.

MATHEMATICS AS A MEANS OF IMPARTING INFORMATION

Obviously the study of mathematics can be of value in a program of democratic education only if it imparts information or stimulates an individual to think. Is the information to be gained from a course in mathematics necessary to an understanding of the world as a student will experience it? Is the study of mathematics of value in developing ability to do reflection-level thinking? Does it help to develop a more harmonious outlook on life? If the study of mathematics is to be essential to an adequate program of education for a democratic society, these questions must be answered in the affirmative.

Mathematicians are, in general, content to point out that our complex civilization is based on practical applications of scientific theories and that a knowledge of mathematics is necessary for complete understanding of modern science. Therefore, they argue, a constantly increasing knowledge of mathematics is essential to understand life as we find it. But those who are less enthusiastic about the practical values to be gained from the study of mathematics point out that the increasing perfection of our machines makes possible their successful operation by unskilled operators. Thus, knowledge of the principles of automotive engineering is not required in order to operate a modern automobile. On the contrary, as automobiles have become more and more complex, knowledge of their inner workings has become less and less needed by a driver. True, an individual who wishes to understand thoroughly the mechanism of his car needs to know a great deal more about science, engineering, and mathematics than was necessary for a similar degree of understanding twenty years ago. On the other hand, however, the average man needs to know much less about them than was necessary twenty years ago. For example, he no longer needs to know how to clean spark plugs, adjust the carburetor, or patch tires. However, it still remains true that, the greater the store of information possessed by a driver about the inner workings of his car, the more intelligently he will be able to operate it.

The same thing holds for an individual's need for a knowledge of mathematics today. For example, an engineer makes extensive use of mathematics in designing today's complicated machines. But, because of the degree of perfection in the final product attained by the expert through use of mathematics, the level of mathematical and technical competence required of the user has declined, as it has in all areas of life. Because of the ever increasing use of mathematics and science by experts, an individual apparently needs to know much less than his ancestors did to satisfy the physical wants of life. Evidently, then, there is a real need for a few persons in our society to study mathematics intensively in order to make possible continuance of the technical advance which makes life so much easier for all of us. But is there any need for the person who has no particular interest in or aptitude for mathematics to study it at all?

In modern times an individual in a democratic society is daily called upon to make decisions, to choose between courses of action, to solve problems which are vital to his welfare but which never troubled his more "skillful" ancestor. To make these decisions intelligently, to solve these problems in a manner most conducive to the continued welfare of himself and of society, he must have a background of experience, actual or vicarious, infinitely greater than was needed a century ago. For example, each voter in a national election is called upon, in actual fact, to pass judgment upon matters of public policy, both social and economic, any phase of which would require a lifetime of study to master thoroughly. Yet policies which may completely alter the way of life of the whole world are determined by the outcomes of these elections. When each individual is called upon to pass judgment in matters so vital, is it not preposterous to suggest that he need know only the few simple skills needed to care for his immediate physical wants?

Is it not more reasonable to suggest that he should constantly strive to add to his store of information regarding the world about him—should constantly strive to achieve greater competence to make these crucial decisions? But no one, however gifted, can possibly possess himself of more than a small portion of the stored-up ideas and information available—the cultural heritage of mankind. For this reason men have sought to formulate general principles which serve to summarize masses of particulars. That is, science represents a studied and persistent effort to abstract general concepts from chaotic masses of specific facts. By means of these general concepts which summarize the accumulated experiences of the race, an individual's power to control his environment is enhanced, and mankind, in turn, is enabled to benefit from the diverse experiences of each person.

Mathematics is the field of human intellectual endeavor in which this process of abstracting generalizations has been most successful. The degree of success in generalization which has been achieved in any field may be measured by the accuracy of prediction made possible by application of the generalizations to specific situations. In mathematics, the degree of precision in prediction attained is so high that it is often taken to be synonymous with certainty, at least in popular opinion. Although in the newer fields of mathematics, such as statistics, the degree of certainty attaching to conclusions reached is not as high as popular opinion would indicate, it is true that in this respect mathematics represents the ideal toward which other sciences strive.

Furthermore, because precision of prediction implies quantitative considerations, knowledge of mathematics is basic to complete understanding of other sciences. This fact is becoming increasingly evident as fields which formerly made little use of mathematics—biology, psychology, and economics, for example—use mathematical concepts increasingly as they approach maturity, i.e., as knowledge in these fields becomes sufficiently extensive and quantitative to make possible precise prediction. Moreover, as Dewey points out, "It is not the mere presence of measurements in quantitative form that yields scientific knowledge, but that particular kind of mathematical statement which can be developed by reasoning into other and more fruitful forms."[20]

Adequate understanding of a growing number of scientific fields can be obtained only with the aid of an extensive background of mathematical knowledge. It appears, therefore, that a knowledge of mathematics is becoming increasingly necessary for intelligent understanding of the world. More than that, mathematics represents one of the oldest and most fruitful fields of human intellectual endeavor. As soon as man developed means of communicating his thoughts to his fellows, he felt a need for giving precise expression to quantitative relationships. This led to the crude beginnings of mathematics. The infant science grew because as man slowly climbed the ladder of civilization he encountered problems which demanded quantitative treatment in greater numbers and complexity. Human curiosity, which impelled individuals to explore the possibilities inherent in particular ideas, was a second factor stimulating the growth of mathematics. From such intellectual explorations of ideas and their consequences has developed the relatively high degree of mastery over material things and forces which characterizes the modern world. From them also stem the insights into nonmaterial values and relationships which differentiate civilized man from the savage. Thus, because of the importance of a knowledge of mathematics as a means of obtaining fuller understanding

[20] John Dewey, *How We Think*, Boston, D. C. Heath and Co., 1933, p. 113.

of our cultural heritage and because of its power to stimulate and sustain intellectual exploration of significant ideas, the study of mathematics can be a significant part of the education of any person who aspires to achieve greater competence to control the course of his future experiences.

THE VOCATIONAL SIGNIFICANCE OF MATHEMATICS

The secondary school has as one of its major functions an obligation to provide students with opportunities to explore widely the major fields of vocational endeavor. Secondary-school students cannot make wise choices of life occupations unless they have adequate opportunity to learn about the various fields open to them and have some basis for assessing their chances of success in them. Since success in a large and ever increasing segment of the professions and vocations of our civilization requires an extensive knowledge of science and mathematics, it would appear that the secondary-school program, if it is to provide adequate opportunities for vocational exploratory experiences, must provide for extensive and serious work in these fields. Mathematics in particular constitutes a fundamental part of the basic preparation for so many occupations and professions that an individual who lacks mastery of its fundamental concepts will find his field of choice of vocation uncomfortably limited. Since he cannot go far in science without knowledge of mathematics, a student without that training will find either that all fields and professions which require a considerable knowledge of the basic sciences are closed to him or that his opportunities in them are greatly restricted. Even in many areas of business and industry, where an individual's competence is not directly dependent on any considerable knowledge of mathematics, the man or woman untrained in this field must rely on the judgments of others or make crucial decisions without taking into consideration factors which may be vital to the situation. If the secondary school is to provide adequate opportunities for exploratory activities, both academic and vocational, the study of mathematics is an essential part of its curriculum. And, if a student is really to explore the possibilities for him in technical and scientific fields, he will find that he needs a rather extensive knowledge of the methods and content of secondary mathematics and a reasonable degree of mastery of its skills and techniques.

MASTERY OF TECHNIQUES AND SKILLS

Concomitant with the obligation to provide exploratory opportunities, the secondary school has a duty to provide opportunities for students who discover, through the exploratory phase of its program or otherwise, a liking for, interest in, and aptitude for mathematics or science to do the necessary preparatory work preliminary to further serious study of these fields. How else can an individual who discovers an interest in mathematics, for example, know whether or not he can hope to work successfully in that field? Moreover, particularly in mathematics and to a lesser degree in the other areas of science, the sequence of required courses is so long that a student who must do any considerable proportion of the preliminary work after leaving high school will be gravely, if not hopelessly, handicapped. Consequently, in most of the professions and vocations requiring knowledge of science or mathematics as a basis for professional preparation it is essential that a prospective student master the basic skills and techniques in secondary school. The implication is that the preparatory needs of this group of students must be considered in planning the total program in mathematics and in science at the secondary level.

MATHEMATICS AND REFLECTION-LEVEL THINKING

Possession of information concerning the world about him and concerning his abilities to deal with the mathematical aspects of his world does not thereby insure that a given individual will be competent to cope with that world. In a democratic society, one grows in competence to participate as a citizen as he gains in competence to discover problems—conflicts and inadequacies—in his outlook and as he gains in power to solve the problems so discovered.

How does one discover conflicts and inadequacies in the world about him? They are discovered when one realizes that, in a given situation, certain observed facts are not consistent with ideas or concepts previously held to be valid. Such a discovery is likely to result when one attempts to formulate a general rule of procedure from consideration of a mass of particular experiences and apparently unrelated bits of information. Or it may arise from an attempt to apply a general concept in a particular situation. The solution of life problems is almost always a matter of either trying to abstract a general principle from a confusion of particulars or testing a general rule of behavior by applying it to a particular situation. Either process involves reflection-level thinking. That is, to solve any problem an individual seeks to abstract from the confusion of details gleaned from past and present experience some general principle which, when applied to the particular situation or perplexity confronting him, indicates a mode of procedure that promises to resolve the difficulty. Or he recognizes a problem when he discovers that, in a given situation, certain observed facts are not consistent with other ideas previously held. A conscious attempt to resolve such difficulties implies a reflective process for, as Dewey has said, reflective thinking is "active, persistent, and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it and the further conclusions to which it tends."[21]

Thus it would appear that the process of solving the problems of life is a reflective process. Then the ability to think reflectively, to recognize and solve crucial, novel problems, is exactly the ability which persons in a democratic society must develop if they are to be competent to accept the responsibilities which citizenship in such a society thrusts upon them. Development in each individual of ability to think reflectively to the highest possible degree would seem, therefore, to be a major concern of a program of education for a democratic society.

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[21] *Ibid.*, p. 9.

It is sometimes asserted that the study of mathematics is especially valuable as a means of developing the ability to do reflective thinking. This assertion is probably true in some cases and may very well be true of a great many individuals. However, as Dewey says, thinking "is the power of following up and linking together the specific suggestions that specific things arouse. Accordingly, any subject, from Greek to cooking, and from drawing to mathematics, is intellectual, if intellectual at all, not in its fixed inner structure, but in its function—in its power to start and direct significant inquiry and reflection. What geometry does for one, the manipulation of laboratory apparatus, the mastery of a musical composition, or the conduct of a business affair, does for another."[22]

The study of any subject will add to one's "power to start and direct significant inquiry and reflection" only to the extent that it adds to his knowledge and understanding of the world and stimulates him to think reflectively about that world. As we have seen, the factual knowledge to be gained from intensive study of mathematics is an important element in gaining clear insights into the physical relationships of the world about us. The techniques and skills of mathematics, properly mastered, add to the precision and facility with which we can consider, compare, and express ideas. Evidently, then, knowledge of mathematics may materially assist an individual to gain a better understanding of the world in which he finds himself and to achieve resolution of the conflicts and perplexities which he discovers there. The study of mathematics obviously affords many opportunities for doing reflection-level thinking. Can it be taught in such manner as to stimulate most individuals to think reflectively about the problems of their daily lives? If it can, it will be of value to most individuals in their struggles to achieve competence to cope with the world as they find it.

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[22] *Ibid.*, pp. 46–47.

CONCLUSIONS

A program of education for a democratic society should afford each individual opportunity to acquire the information and to develop the attitudes and habits of thinking needed to cope with a changing world—to achieve competence to make wise decisions in novel and crucial situations. A person who develops into an independent learner always actively seeking to extend his store of information about his world and to broaden and harmonize his insights into its relationships becomes thereby progressively more competent to cope with that world.

The subject matter of a program of democratic education should be, therefore, material which will (1) add to one's knowledge of the world, (2) help to harmonize his outlook on that world, and (3) stimulate him to think reflectively about it and its problems. Mathematics is a fruitful source of information about the world. The methods of mathematics are essentially the techniques found most useful in solving problems in any situation. The study of mathematics, then, because it is a means by which individuals may acquire information essential to competence to cope with the world, because it affords practice in ways of thinking which are useful in gaining a measure of control over one's environment, and because it can be so taught as to help an individual learn to think reflectively, can make significant contributions to a program of democratic education.

It must be remembered, however, that the informational, cultural, and utilitarian values to be gained from a study of mathematics represent potential goods, to be realized only through effective use of mathematical subject matter. They are not virtues inherent in the subject matter itself. Obviously, such desirable potentialities cannot be fully realized if the emphasis in mathematics teaching is on techniques and manipulations useful only to students of advanced mathematics. Neither can they be realized if the emphasis is exclusively on the utilitarian applications of mathematics.

In reality, the controversy over the place and function of mathematics in a program of modern democratic education is a challenge to teachers of mathematics to devise and institute teaching procedures which will make the most effective use of the rich heritage from the past that is mathematics. To meet this challenge, teachers must cease to teach mathematics as if they assumed that every individual in their classes would become a research mathematician. Too many students never see the beauty of the forest that is mathematics because they are exclusively concerned with the trees that are techniques or details. How many students never realize that algebra is really generalized arithmetic? How many college students of mathematics never discover that analytic geometry is the meeting ground of algebra and geometry? It is a teacher's responsibility to see that students are given adequate opportunity to master such important concepts of mathematics. It is the function of the teacher to make clear the implications of mathematical concepts to life. There is nothing dull or boresome about the materials taught in mathematics classes. If the study of mathematics is dull, can the fault be in the way that study is carried on?

However, the student is the one finally responsible for gaining the utmost in value from his study of mathematics, or of any other subject. Mathematics is not as difficult to understand as is often claimed by unsuccessful students. On the other hand, it cannot be mastered without persistent, sustained effort on the part of the learner. In school, as in life, one tends to get out of an activity values and satisfactions approximately proportional to the effort one puts into it. A student will gain from the study of mathematics, in the way of increased competence in living, in proportion to the effort, thought, and care that he puts into it.

It seems reasonably evident that, given competent instruction, any normal individual may, by conscientious study, profit greatly from a study of mathematics. It can make unique contributions to one's understanding of and ability to cope with his environment and so be an essential part of any adequate program of education for a democratic society. We are forced to admit, however, that the subject has not always been taught so as to realize its educative potentialities. The chapters which follow present a theory of teaching mathematics which seems to give promise of achieving a higher degree of success in this regard.

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PART II

Teaching Problems in Secondary-School Mathematics

Chapter 6

Demonstrative Geometry

According to J. W. A. Young, "Geometry, perhaps more than any other subject of secondary school mathematics, offers opportunity for attaining all the ends of the teaching of mathematics." He continues:

> It gives ample occasion for exact reasoning, for real induction applied to very simple data, for correlation with other work, with drawing, geography, and the physical sciences as well as with algebra, for exercise of the space intuition, for practical applications, for drill in numerical computation, for training in habits of neatness and exactitude, and for cultivation of the powers of precise thought and accurate expression. If from among all these a selection were to be made, development of the space intuition and training to active and careful reasoning might be regarded as pre-eminently the function of the teaching of geometry.[1]

If, therefore, the ultimate aim of education is to develop an independent learner—that is, one who has the ability to do reflective thinking, then the study of geometry would seem to be an essential part of secondary education.

Unfortunately, as Blackhurst has so aptly said, "Both the content and method of geometry have through the long ages been such that almost every conceivable obstacle has been placed across any path which might lead to a consideration of the reasoning which is being done."[2] For, as Fawcett points out, although verbal allegiance is paid to the idea that the chief contribution of the study of demonstrative geometry to secondary education is the training it gives in exact reasoning, the emphasis in teaching is placed elsewhere. Fawcett says that "actual classroom practice indicates that the major emphasis is placed on a body of theorems to be learned rather than on the *method* by which these theorems are established. The pupil feels that these theorems are important in themselves and in his earnest effort to 'know' them he resorts to memorization. The tests most commonly used emphasize the importance of factual information, and there is little evidence to show that pupils who have studied demonstrative geometry are less gullible, more logical and more critical in their thinking than those who did not follow such a course."[3]

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[1] J. W. A. Young, *The Teaching of Mathematics in the Elementary and the Secondary School,* New York, Longmans, Green and Co., 1925, p. 259.

The reasons for this discrepancy between theoretical aim and actual practice are not difficult to discover. In the first place, until recently textbooks were modeled closely on the pattern set by Euclid. And Euclid, whatever his virtues as a writer of textbooks for mature minds, did not write a book well adapted to the needs of adolescents. As long as the high school was highly selective, students with high academic aptitude could memorize proofs and so "get by" and some of them were even able to appreciate the logic employed. Abraham Lincoln, we are told, studied geometry to develop his ability to reason logically. But Lincoln was a mature man at the time he took up the study of geometry, and few secondary-school students who studied the formal geometry of Euclid were able, like Lincoln, to recognize its value as a means of developing their ability to reason logically. So, as secondary-school enrollments began to climb, the number of failures in geometry also began to increase rapidly and educational theorists and harassed administrators began to question the educative values of the subject.

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[2] J. Herbert Blackhurst, *Euclidean Geometry, Its Nature and Use,* Des Moines, Iowa, Garner Publishing Co., 1947, p. 100.

[3] Harold P. Fawcett, *The Nature of Proof,* Thirteenth Yearbook of the National Council of Teachers of Mathematics, New York, Bureau of Publications, Teachers College, Columbia University, 1938, p. 1.

Moreover, the tendency of teachers to teach as they had been taught perpetuated the uncritical, *memoriter* type of geometry teaching. Given a subject, then, in which the accepted textbooks were patterned after Euclid and the teaching methods still followed the plan in vogue when the doctrine of formal discipline was generally accepted, and a clamor for studies of immediate practical utility, it is not surprising that demonstrative geometry came close to being discarded along with Greek and the concepts of faculty psychology.

There is, of course, little in the traditional secondary-school course in demonstrative geometry to warrant its being retained in the curriculum if the criterion of immediate utility is rigidly applied. The information afforded by the subject matter relative to spatial relationships is pretty largely taught intuitively in the elementary grades and in the ninth-grade course in general mathematics. So, if we accept Wrinkle's contention that demonstrative geometry has no cultural or disciplinary values,[4] it is logical to conclude that, except for the few who need the course to meet college-entrance requirements, there is no place for demonstrative geometry in the secondary-school curriculum.

PRIMARY AIM OF INSTRUCTION IN GEOMETRY

But if we agree that the fundamental aim of education in a democratic society is to develop an individual's ability to do critical, reflection-level thinking about the problems which will confront him in his daily life, we need to give consideration to Upton's contention that "the reason we teach demonstrative geometry in our high schools today is to give pupils certain ideas about the nature of proof."[5] Upton insists that the purpose of teaching geometry is not primarily to acquaint pupils with geometric facts or theorems or with the applications of these facts or theorems to everyday life but that "our great aim in the tenth year is to teach the nature of deductive proof and to furnish pupils with a model of all their life thinking."[6] Blackhurst argues that when we search for subject matter with which to study the reasoning process and to "provide for experience with reasoning, we find logical geometry to be not only the best object matter for this purpose but object matter which could have no other possible educational purpose."[7] It is important to note that, if Upton and Blackhurst are correct in their contentions, the subject matter of demonstrative geometry is only of secondary importance. In fact, as Blackhurst implies, one of the advantages of using geometrical subject matter as a means of studying the reasoning process is that we have no prejudices about it—our emotions are not involved—and so we can center our attention, not on the conclusions reached, but on the manner in which they were reached.[8] More than that, the subject matter of geometry is relatively simple and well organized, and the relationships between basic assumptions and the conclusions reached can be readily demonstrated.

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[4] William L. Wrinkle, *The New High School in the Making*, New York, American Book Co., 1938, pp. 180–181.

[5] C. B. Upton, "The Use of Indirect Proof in Geometry and Life," *The Teaching of Geometry*, Fifth Yearbook of the National Council of Teachers of Mathematics, New York, Bureau of Publications, Teachers College, Columbia University, 1930, pp. 131–132.

The fundamental reason for teaching demonstrative geometry, therefore, is that it is the most suitable means to familiarize pupils with that type of thinking which Keyser has called "autonomous" or "postulational" or "If-Then" thinking, a type "distinguished from all others by its characteristic form: if so-and-so, then so-and-so."[9] Betz summarizes this argument for continuing to teach demonstrative geometry in the secondary school as follows:

> Euclid's *Elements of Geometry* . . . was the first instance in human history of autonomous thinking. . . . This colossal achievement remained unique for many centuries, but it has since become the prototype of *all* valid scientific investigations. And thus geometry has given to the world its one reliable method of thinking. *Postulational thinking* is now seen to be of basic importance in any field of human endeavor. Even a cursory

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[6] *Ibid.*

[7] Blackhurst, *op. cit.*, p. 102.

[8] *Ibid.*, pp. 102–103.

[9] Cassius J. Keyser, *Thinking About Thinking*, New York, E. P. Dutton and Co., 1926, pp. 18 ff.

analysis shows that *demonstrative geometry offers the simplest and most convenient introduction to postulational thinking that has yet been devised.* Hence it may be claimed that the teaching of demonstrative geometry is not only justifiable, but absolutely essential, because of its permanent devotion, in a singularly pure and significant form, to the one procedure that promises valid conclusions on the basis of clearly formulated assumptions. [Italics in the original.][10]

If this point of view be accepted, the *memoriter* type of geometry teaching so often encountered is completely inappropriate. Unless geometry can be taught so that it is effective as a means of increasing a pupil's ability and willingness to do reflective, critical thinking, all that Wrinkle and the others who object to retaining the study of demonstrative geometry in the curriculum of secondary education have said about its nonfunctional character is valid.

Can geometry be taught so as to develop the ability to think reflectively? Can it be made functional in non-school situations? Fawcett has given us an account of an attempt to achieve these aims.[11] He took as his objective in the experiment he describes the twofold goal of leading pupils to understand the nature of deductive proof through a study of geometric situations and at the same time to develop the ability to generalize this concept so that transfer to nongeometric situations would be made.

THE NATURE OF GEOMETRIC PROOF

The starting point in Fawcett's experiment was, obviously, to decide just what it is that a pupil has learned when he understands the nature of geometric proof. Fawcett assumed that the pupil who understands the nature of proof understands:

1. The place and significance of undefined concepts in proving any conclusion.
2. The necessity for clearly defined terms and their effect on the conclusion reached.

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[10] William Betz, "The Transfer of Training, with Particular Reference to Geometry," *The Teaching of Geometry*, Fifth Yearbook of the National Council of Teachers of Mathematics, New York, Bureau of Publications, Teachers College, Columbia University, 1930, p. 151.

[11] Fawcett, *op. cit.*

3. The necessity for assumptions or unproved propositions.
4. That no demonstration proves anything that is not implied by the assumptions.[12]

But the problem of a geometry teacher is not only to bring his pupils to an understanding of the nature of proof but also to stimulate them to make actual use of this concept in their lives outside of mathematics classrooms. It is of little value to an individual to understand thoroughly what is necessary to prove a proposition unless this knowledge leads him to clarify his own thinking and to be critical of new ideas presented to him. Fawcett says:

> The real value of this sort of training to any pupil is determined by its effect on his behavior, and for purposes of this study we shall assume that if he clearly understands these aspects of the nature of proof his behavior will be marked by the following characteristics:
>
> 1. He will select the significant word or phrases in any statement that is important to him and ask that they be carefully defined.
> 2. He will require evidence in support of any conclusion he is pressed to accept.
> 3. He will analyze that evidence and distinguish fact from assumption.
> 4. He will recognize stated and unstated assumptions essential to the conclusion.
> 5. He will evaluate these assumptions, accepting some and rejecting others.
> 6. He will evaluate the argument, accepting or rejecting the conclusion.
> 7. He will constantly re-examine the assumptions which are behind his beliefs and which guide his actions.[13]

It is the task of the teacher, then, to utilize the subject matter of geometry to develop in his pupils the habits of reasoning suggested above. Geometry is better adapted to this purpose than any other subject because it does exhibit the logical organization devoid of emotional content needed. But it is necessary to go beyond geometric material to insure that a pupil will recognize that he can apply the reasoning techniques he has learned to life situations outside the classroom. As we have seen (see pp. 125 ff.), if there is to be transfer of training from classroom to life situations, an individual must recognize both that an opportunity for transfer of training exists and that his immediate purposes will be served by making the transfer. It is not difficult for students to discover opportunities to transfer geometric concepts to problem situations involving spatial relationships. They are not likely to recognize occasions for using the reasoning techniques of geometry in nongeometric situations unless such applications have been part of their classroom experiences in geometry. For, as Wheeler says, "No transfer will occur unless the material is learned in connection with the field to which transfer is desired."[14] And Fawcett adds, "Transfer is secured only by training for transfer and teachers can no longer expect that the careful study of ninety or more geometric theorems will alone enable their students to distinguish between a sound argument and a tissue of nonsense."[15]

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[12] See *ibid.*, p. 10.

[13] *Ibid.*, pp. 11–12.

HOW CAN GEOMETRY BE TAUGHT FOR TRANSFER?

How can a teacher utilize the ninety or more theorems that Fawcett mentions to develop in his students increased ability to do critical thinking? In particular, how can the study of geometry with its noncontroversial subject matter serve to develop increased ability to think clearly on controversial questions, as in politics, for example?

The way in which logical structure is developed may afford a suggestion as to a workable plan of procedure. The basis of any logical system is a few concepts which are not defined but which are definite and unambiguous. These are the foundation stones, as it were, on which the system is built. Obviously, it is well to keep the number of these undefined concepts as small as possible and to define all other, more complex terms and concepts utilized by means of these simpler, basic ideas. Certain statements on the relationships existing between these basic concepts are also needed. These we must agree to accept without proof. The system is then built up by a process of deducing other relationships from these basic ones. Proof consists simply in showing that the conclusions reached are necessary consequences of the propositions originally accepted.

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[14] Raymond H. Wheeler, *The New Psychology of Learning*, Tenth Yearbook of the National Council of Teachers of Mathematics, New York, Bureau of Publications, Teachers College, Columbia University, 1935, p. 239.

[15] Fawcett, *op. cit.*, p. 13.

It would appear, therefore, that one might well start the study of demonstrative geometry with an examination of the basic concepts and relationships on which the theory of the subject rests. If it can be shown that the procedure by which relationships are deduced and propositions proved in geometry will likewise be fruitful in other, nonrelated fields, then the study of this procedure will be found to have potential transfer value.

Fawcett had his pupils begin the study of geometry with the problem of clear definition—not of geometric terms but of ideas and concepts familiar to them in their everyday living. Words such as *restaurant, capitalist, farm, school,* and *student,* to give only a few, afford excellent examples of terms freely used yet vaguely defined. It is not at all difficult to show how acceptance of one definition in contrast to another equally plausible will alter the conclusion to be drawn in a given situation. From study of the difficulties encountered in framing acceptable definitions of terms in common use it is not difficult to lead students to see the advantages of using concepts from geometry as a basis for studying the type of thinking involved in arriving at reasoned conclusions. Nor is it difficult to agree on certain statements—postulates—which will be accepted without proof. Fawcett found, for example, that his students readily agreed to accept, as undefined but unambiguous, terms such as *point, line, plane,* and *angle.* Similarly, they worked out several statements of relationships between these fundamental concepts which they agreed to accept without proof.

It is possible, of course, to develop the whole of the typical secondary-school course in demonstrative geometry from a relatively few undefined concepts and unproved statements. But geometry is studied in the secondary school for the purpose of gaining power in the use of its characteristic method rather than to determine how few assumptions need to be made to prove its theorems. Beginners, especially, lose interest if they must take time to prove propositions which seem self-evident to them. So it will usually be found wise to assume many more theorems than are absolutely necessary and to concentrate on those theorems whose proof is not readily evident to a student.

THE SYLLOGISM

Once a set of basic assumptions has been selected, the next step in developing a logical system is to see what conclusions can be drawn from these assumptions. Possibly the simplest form of such a system is the syllogism. Fundamentally, of course, a syllogism consists of three statements: (1) a statement of a general law or principle called the major premise; (2) a statement calculated to show that the general principle set forth in the major premise is applicable to the matter in hand—the minor premise; and (3) a statement of the conclusion which must follow. It is not difficult to obtain numerous examples dealing with all sorts of topics in the form of the classic:

> All men are mortal.
> Socrates was a man.
> Therefore Socrates was mortal.

Time spent in experimenting with the interchange of the three statements of various syllogisms will prove most helpful in making clear the concept of proof as consisting in showing that a conclusion necessarily follows acceptance of the two premises and, also, that tacit assumptions read into a situation may lead to fallacious conclusions. The syllogism also affords an excellent means of illustrating the idea of the converse of a proposition and of showing that the converse of a valid proposition is not necessarily valid.

Ulmer has shown that everyday conversations and advertisements are fruitful sources of syllogisms in which one is asked to accept certain tacit assumptions which he probably would not accept if they were openly stated.[16] A few of the examples he lists will be sufficient to indicate how material from such sources may be used, first to give a clearer understanding of the nature of proof, and second to afford the student an opportunity to transfer what he has learned in geometry to nongeometric situations. Ulmer suggests:

> Next, have the class consider examples in which only a part of the argument is given and have them build the rest of it. For instance, call their attention to the following statement, which is a common kind: "This is a Spaulding tennis racket so it must be a good one." Ask what assumptions would have to be made—what statement would have to be taken for granted—in order to insure the conclusion contained in the above statement. After this question has been answered, the class should put the statement in the following form:
>
> (1) All Spaulding tennis rackets are good rackets.
> (2) This is a Spaulding tennis racket.
> (3) This must be a good racket.
>
> Another example is: "this must be good music, it's grand opera." The class should first decide what the conclusion is, then find an assumption which would make this conclusion necessary. The argument should finally be stated as follows:
>
> (1) All grand opera is good music.
> (2) This is grand opera.
> (3) This must be good music.
>
> . . . An example taken from advertising matter is: "Only Pepsodent Tooth Powder contains Irium." There is no conclusion stated but undoubtedly the advertiser wanted us to conclude—that is, he wanted us to have the feeling somehow—that Pepsodent is the best tooth powder. After agreeing on the conclusion that we were meant to draw, the class should find the assumption it would be necessary to make in order that this conclusion must follow.[17]

Many modern textbooks in geometry also contain sections on the various forms and uses of the syllogism.

Unfortunately, a geometric theorem is usually given in a form which does not clearly reveal its syllogistic character. However, merely shifting the order of statements in a relatively simple geometric proof will serve to reveal the way a series of syllogisms may be utilized to establish the validity of geometric propositions. Suppose we are to prove that the bisector of the vertex angle of an isosceles triangle bisects the base of the triangle. The traditional style of proof is as follows:

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[16] Gilbert Ulmer, "Some Suggestions for Teaching Geometry to Develop Clear Thinking," *Kansas Studies in Education*, Vol. 2, No. 7, Lawrence, Kansas, University of Kansas Publications, 1942.

[17] *Ibid.*, pp.6–7.

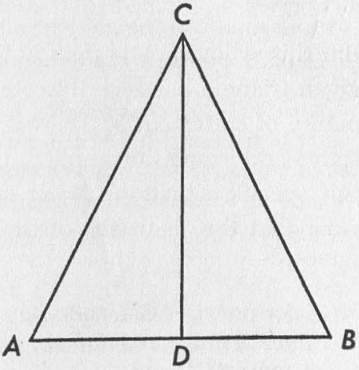

FIG. 1

THEOREM: *In an isosceles triangle, if a line segment bisects the vertex angle, then it also bisects the base of the triangle.*

GIVEN: The isosceles triangle ABC (Fig. 1) with CD bisecting the vertex angle ACB.

TO PROVE: $AD = DB$.

PROOF:

1. $AC = CB$	1. By hypothesis ($\triangle ABC$ is isosceles)
2. $CD = CD$	2. Identical
3. $\angle ACD = \angle BCD$	3. CD bisects $\angle ACB$
4. $\triangle ACD \cong \triangle BCD$	4. If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent.
5. $AD = DB$	5. Corresponding parts of congruent triangles are equal.

Let us examine the first four statements of this proof for a moment. Is one of them essentially a statement of a general truth or law? Is there a statement or statements showing that the general law cited is applicable in the present instance? What conclusion is reached by applying the general principle in this specific instance? It is at once apparent that the first four statements may be recast in syllogistic form as follows:

1. Major premise: If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, then the triangles are congruent.

2. Minor premise: Since $AC = BC$, $CD = CD$, and $\angle ACD = \angle BCD$, triangles ACD and BCD have two sides and the included angle of the one equal respectively to two sides and the included angle of the other.
3. Conclusion: Triangle ACD is congruent to triangle BCD.

But this syllogism still does not clinch the fact that the base of the given triangle has been bisected. In the formal proof, what is the nature of statement 5? Does it cite a general principle? Is there a statement, express or implied, that this general principle applies to this specific situation? What conclusion is reached? It should be clear that this final step of the formal proof implies a second syllogism:

1. Major premise: Corresponding parts of congruent triangles are equal.
2. Minor premise: AD and DB are corresponding parts of the congruent triangles ACD and BCD.
3. Conclusion: $AD = DB$.

It will sometimes be helpful to have students recast the statements of the proofs of theorems into syllogistic form as a means of showing that a geometric proof consists of a sequence of logically related statements demonstrating that the proposition to be proved is the necessary consequence of prior assumptions and proofs previously established.

INDUCTIVE THINKING AS A PART OF PROOF

It is essential that a teacher of geometry recognize that the formal proving of a theorem is only part of the total reasoning process involved. The steps which constitute a reflective process are:

1. Recognition that a problem exists and accurate formulation of that problem—plus a compelling reason for seeking a solution.
2. Collection and organization of available information relative to the problem under consideration.
3. Formulation of a hypothesis for solution of the problem.
4. Testing of this proposed solution. If it turns out that the hypothesis was correct, that the method proposed does provide a satisfactory solution to the difficulty, the process ends. If the test reveals

that the proposed solution is not entirely satisfactory, then further steps must be taken to resolve the difficulty.

5. If, after testing, the proposed solution is found to be inadequate, it will be necessary to reformulate the problem in the light of the new data obtained or the new insight into the real nature of the difficulty gained from the test. If necessary, additional information may be gathered, the original hypothesis modified or discarded, and a new one formulated and tested. This process of formulating and testing other hypotheses continues until the problem is solved or abandoned as insoluble.

Actually, in this process, steps 1 and 2 are essentially inductive. Step 3 is a matter of stating a proposition to be proved and step 4 may be a deductive proof. In short, the formal proof of a theorem in geometry is not a complete reflective process. It involves only steps 3 and 4. The preliminary analysis leading to formulation of the proof is an essential part of the total reasoning process.

Suppose we consider the proof of the following theorem:

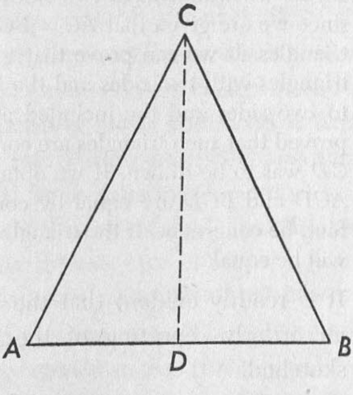

FIG. 2

If a triangle is isosceles, then the angles opposite the equal sides are equal.

GIVEN: The isosceles triangle ABC, with $AC = BC$ (Fig. 2).

TO PROVE: $\angle BAC = \angle CBA$

CONSTRUCTION: Construct the bisector of $\angle ACB$ meeting AB in D.

PROOF:

1. $AC = BC$	1. Given
2. $CD = CD$	2. Identical
3. $\angle DCA = \angle DCB$	3. CD bisects $\angle ACB$
4. $\triangle ADC \cong \triangle DBC$	4. Two triangles having two sides and the included angle of one equal to the corresponding two sides and included angle of the other are congruent.
5. $\angle BAC = \angle CBA$	5. Corresponding parts of congruent triangles are equal.

Actually, this proof as given consists of only the final steps taken in the solution of the original problem. It is but an orderly presentation of the results of a thinking process which began with the statement of a problem about relationships existing between angles opposite the equal sides of isosceles triangles. The total thinking process was somewhat as follows:

Theorem: If a triangle is isosceles, then the angles opposite the equal sides are equal.

1. Suppose we have a triangle ABC (see Fig. 2) in which side AC is equal to side BC. What is the relationship between $\angle BAC$ and $\angle CBA$? We are asked to prove that they are equal.

2. One of the simplest ways of proving two angles equal is to show that they are corresponding parts of congruent triangles. Now, if a line could be drawn from C meeting side AB in some point D, that line would divide $\triangle ABC$ into two triangles, $\triangle ADC$ and $\triangle BDC$. In these two triangles we would have two sides of one equal to two sides of the other since we are given that $AC = BC$ and the line CD is identical in the two triangles. If we can prove that $\angle ACD = \angle BCD$, then we will have two triangles with two sides and the included angle of one equal respectively to two sides and the included angle of the other and we have already proved that such triangles are congruent. But we did not specify how line CD was to be drawn. If we obtain CD by bisecting $\angle ACB$, then the $\angle$s ACD and BCD are equal by construction and the two triangles will, in fact, be congruent. If the triangles are congruent, then $\angle BAC$ and $\angle CBA$ will be equal.

It is readily evident that the formal proof given previously is but an orderly reporting of the results of the thinking process just sketched.

TYPES OF PROOF

The proof described above is direct. That is, it is possible to show directly that certain lines and angles are equal, hence that the two triangles formed when line CD is constructed are congruent. Proofs of this type are, in general, readily understood by students.

A second type of proof, that of superposition, is needed to prove the basic theorems on congruence of triangles. This involves the concept of picking one figure up and placing it on another in such manner that specified parts coincide. The procedure is, of course, based on the assumption that geometric figures are rigid bodies which can be moved about in space without change in size or shape. Students, as a general rule, do not have great difficulty with this concept, particularly if care is used in indicating just how the one figure is to be placed on the other and which parts are to be made to coincide. It will usually be found desirable, the first time or two that the method is used, to use dividers to show the way in which line segments may be transferred, to construct equal angles, and so on. If this is done, the final step is merely one of showing by direct proof that the two figures do coincide exactly in every part. Smith believes, however, that proof by superposition should be avoided if and whenever possible.[18]

Actually, since the aim in teaching demonstrative geometry at the secondary level is to emphasize a method of thinking rather than to develop a rigorous, logical sequence of theorems, little is gained by insisting on formal proof that two triangles having two sides and the included angle or two angles and the included side of one equal respectively to the corresponding parts of the other are congruent. These facts can readily be established informally and the techniques of formal proof be reserved for less obvious theorems.

Most of the theorems on congruent triangles, other than the two just mentioned, can be proved directly if these two are established intuitively or taken as assumptions. However, although direct proofs are usually more convincing to students, some important theorems cannot be proved directly or by superposition. For example, the proposition, if two lines in the same plane are situated so that when cut by a transversal the alternate interior angles are equal, cannot be proved by either method. It must be proved by the indirect method.

INDIRECT PROOFS

Essentially an indirect proof consists in stating all possible relationships existing in a situation and then eliminating all but one of these possibilities by showing that they are incompatible with the facts of the case and that, therefore, the remaining possibility must be a correct statement of the existing relationship. As Upton has indicated, pupils and teachers both tend to dislike and distrust the indirect method of proof.[19] They feel that it is not a straightforward attack on a problem, that it "beats around the bush" and really does not prove anything. In short, to many teachers and students of geometry, indirect proofs are not convincing. Yet indirect proof may be just as conclusive as direct proof. And, not only in geometry but in life situations as well, an indirect approach may be the only feasible way of establishing the validity of a proposition. In law, for example, it is often impossible to establish directly the guilt or innocence of an accused person. Criminals do not ordinarily operate in the presence of witnesses. But such well-publicized police methods as use of fingerprints and of rifling marks on bullets merely provide links in a chain of indirect proof which leads to conviction. Circumstantial evidence is often the only kind possible, and conviction of a criminal by circumstantial evidence is merely a matter of establishing his guilt by the method of indirect proof. On the other side of the picture, the familiar alibi is also a form of indirect proof—"I couldn't have done it; I wasn't even there." A physician must often diagnose an illness by eliminating possibilities. So must an auto mechanic or a radio repairman. There will be occasions when all of us must arrive at conclusions by eliminating possibilities —by reducing all but one of the proposed or possible solutions to a problem to absurdities. Upton quotes Jevons, the great logician, as saying that nearly half the judgments we form are based upon indirect reasoning.[20]

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[18] David Eugene Smith, *The Teaching of Geometry*, Boston, Ginn and Co., 1911, p. 169.

If, therefore, the indirect method of proof is of such great practical importance, it would seem that a student of demonstrative geometry has failed to achieve maximum gain in reasoning power from his study of geometry if he does not gain therefrom an understanding of the indirect method of proof and ability to use it effectively.

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[19] Upton, *op. cit.*, pp. 102–133.

[20] *Ibid.*, p. 104.

The logical basis for an indirect proof is found in the concept of contradictory propositions. Two propositions are contradictory when they are such that if one is true the other must be false. Obviously, therefore, if one of two contradictory propositions is shown to be false, the other is thereby shown to be true. A second principle of logic utilized in indirect proofs is that, if a process of correct reasoning leads to a false conclusion, then the premises on which the conclusion is based must be false.

Possibly the most satisfactory way to introduce the idea of indirect proof is in connection with the problem of proving a specific theorem. Suppose, for example, that the method of indirect proof is to be used for the first time in connection with the theorem: *If two straight lines in the same plane are both perpendicular to the same straight line, then the two given lines are parallel.* Brief investigation will serve to show that neither the method of direct proof nor the method of superposition will satisfactorily establish the desired relationship. However, suppose we ask, "What are the possible relationships which may exist between two straight lines in a plane?" It will not be difficult to establish that they must either be parallel or intersect in a point. Hence it is easy to set up the two contradictory propositions:

1. Lines *a* and *b* are parallel.
2. Lines *a* and *b* meet in a point *P*. (See Figs. 3 and 4.)

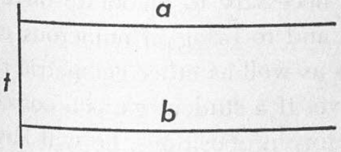

FIG. 3

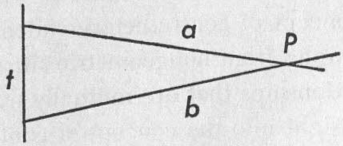

FIG. 4

Obviously, proposition 1 is what we are trying to prove. But what would be the consequences of accepting proposition 2? What known relationship exists between lines *a* and *t*? Between lines *b* and *t*? If *a* and *b* are both perpendicular to *t* and also meet at *P*, what conclusion must necessarily follow? Is it possible that there can be more than one line in a plane from a given point perpendicular to a given line? Evidently, then, our assumption that proposition 2 correctly describes the relationship between lines *a* and *b* has led us to a false conclusion. But there was no error in the reasoning which led to this false conclusion. Where, then, must the error lie? Evidently the premises on which this false conclusion was based must not be tenable. But if the proposition that *a* and *b* meet at *P* is false, then the contradictory proposition that *a* and *b* are parallel must be true since it is the only other possible relationship for lines *a* and *b* under the given conditions.

Students will probably not experience difficulty in following this line of reasoning but very probably they will not feel that they have proved the lines parallel. This lack of a feeling of assurance is, of course, the fundamental objection to the indirect method as a method of proof. The reason for it may be lack of a real understanding on the part of students of the two fundamental concepts of logic employed. If so, a logical next step is to go back over the proof and this time to call attention to the reasoning processes employed rather than to the geometric relationships involved. Just as the idea of proof as consisting in showing that a given conclusion was the necessary consequence of stated assumptions was developed through the study of syllogisms applied to nongeometric as well as to geometric material, so it will be necessary to elaborate on the concept of contradictory statements and to bring in numerous examples from nongeometric situations as well as other geometric relationships that are mutually exclusive. If a student gains a correct insight into the concept of contradictory propositions, he will have no difficulty in recognizing that to show one of the only two possible statements about a relationship to be false does, in fact, prove the other to be true.

Once we have the concept of contradictory propositions firmly established, it will not be difficult to extend this concept to situations where there are more than two possibilities, as in the theorem: *If two angles of a triangle are unequal, then the sides opposite these angles are unequal and the greater side is opposite the greater angle.*

Here we have $\triangle ABC$ (Fig. 5) with $\angle A < \angle B$. What are the possibilities concerning the relationships of sides a and b? They are readily seen to be:

1. $a = b$.
2. $a < b$.
3. $a > b$.

It is evident that only one of these possibilities can be a true statement of the relationship existing between a and b. A student who really understands the concept of contradictory propositions will have no trouble in seeing that, if the first and third relationships are eliminated as possibilities, the validity of the second will have effectively been established.

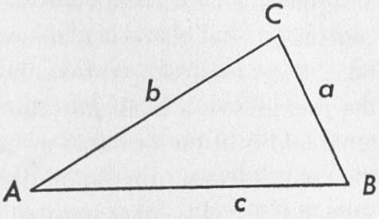

FIG. 5

Similarly, practice in showing that correct reasoning processes based on false premises must inevitably lead to false conclusions will go far toward eliminating any remaining doubts the student may have as to the validity of an indirect proof.

Students frequently have trouble with indirect proofs because they find it hard to assume the contrary to what they are asked to prove. As Upton points out, this difficulty can be avoided by making the contradictory proposition a hypothetical statement of what might be true rather than a positive statement of fact.[21] Then the consequences of accepting the hypothesis can be stated using the verb *would be* instead of *is*. Thus, in the example used above (referring to Fig. 5), a student might object to saying that side a *is* equal to side b when the theorem definitely implies that it is not. But suppose care is taken to make the statement in the form, "If a

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[21] *Ibid.*, p. 114.

were equal to b, then $\angle A$ *would be* equal to $\angle B$." Have we not then emphasized the fact that a is not equal to b? Upton ascribes the suggestion for use of this device to De Morgan but points out that Euclid actually used it in his indirect proofs.[22]

Finally, after the total concept of indirect proof is well established, a student should have sufficient practice in using the method to gain some facility and confidence in his ability to use it. As indicated above, the method has wide and frequent applications to life problems. Therefore, a student should have considerable practice in applying it to problems growing out of daily experiences. After all, only when he sees that a method will work on problems which seem real to him—and only problems growing out of his own experiences seem real to him—does he actually see an opportunity for making a transfer of training, for using what has been learned in school to control a life situation.

The method of indirect proof, or proof by reduction to an absurdity, as it is sometimes called, is the best if not the only way of proving certain important theorems in geometry and in other branches of mathematics as well. It is also widely used outside the classroom. It is, therefore, an essential part of the demonstrative geometry course. And if the fundamental aim of a geometry course is to teach pupils to think clearly, the course will fail in its purpose unless it does provide for developing sound understanding of the techniques and potentialities of the indirect method of proof.

LOCUS

The idea of a geometric locus is another concept which is relatively simple but often proves somewhat elusive to a student. Breslich believes that study of this topic is usually deferred until late in the geometry course because it is regarded as difficult by teachers.[23] He feels that a pupil will have had actual experiences with locus problems long before he studies geometry and that examples illustrating the idea of a locus as the path of a moving point may be brought in very early in a student's work in mathematics. Minnick says that the average pupil has his first contact with the idea of locus in the eighth grade and that the concept is well developed experimentally before a course in demonstrative geometry is reached.[24] Smith, on the contrary, says that the idea of locus is usually introduced after students have studied the theorems on congruent triangles, parallel lines, perpendiculars, quadrilaterals, and the angles of polygons. He thinks it must be deferred until then because "It is too abstract to be introduced any earlier, although authors repeat the attempt from time to time, unmindful of the fact that all experience is opposed to it."[25]

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[22] *Ibid.*

[23] Ernst R. Breslich, *Problems in Teaching Secondary School Mathematics*, Chicago, University of Chicago Press, 1931, p. 299.

However, the difference of opinion over the proper time to introduce the concept of geometric locus is not as sharp as might be inferred from these statements. Smith has definitely in mind the formal treatment of the topic in demonstrative geometry, and Breslich cautions that, even though locus concepts are studied intuitively early in a pupil's mathematical experience, a formal study of the topic should not be attempted too soon.[26] It appears, therefore, that the concept of locus is simple enough to be introduced intuitively to relatively young students but is so difficult a topic in demonstrative geometry that its formal study must be deferred until fairly late in the course.

There are probably two reasons for this apparent inconsistency. In the first place, the term *locus*, with its Latin plural *loci*, is not part of an average student's vocabulary. And, unlike the similar *radius* and *radii*, there is no simple, readily understood definition of it. The obvious statement that *locus* is the Latin word for *place* or *position* is really not very helpful since neither *place* nor *position* is exactly equivalent in meaning to the term *locus* as used in geometry. In the second place, a locus is commonly described as the path of a moving point which must obey certain fixed requirements as it moves. This is the way Breslich proposes to introduce this concept early in the mathematics program. Yet students have not previously had experience with the idea of a line as the path of a point in motion. The points and figures studied before have been static. Moreover, the definition of locus commonly given in geometry textbooks implies the static concept of locus.

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[24] J. H. Minnick, *Teaching Mathematics in the Secondary Schools*, New York, Prentice-Hall, Inc., 1939, p. 213.

[25] Smith, *op. cit.*, p. 198.

[26] Breslich, *op. cit.*, p. 299.

By one definition, then, a geometric locus is a figure containing all points, and only those points, which satisfy certain given conditions. This definition obviously implies that the points remain fixed and that there may be many such points, each of which satisfies the given conditions. The second definition presents a locus as the path of a single point moving according to certain specified conditions. By the first definition, the bisector of an angle would be presented as the location of all points which were equally distant from the sides of the angle whereas the second definition would present it as the path of a single point which moves so that it is always equally distant from the sides of the angle. Since both concepts may be useful on occasion, both should be presented and developed. The concept of locus as the path of a moving point—the dynamic concept—may be illustrated by such familiar examples as the path of a ball thrown through the air, of a stone tied to a string and whirled about a point, or of a pendulum of a clock. The static concept may well be introduced by means of such simple constructions as the perpendicular bisector of a line segment and the bisector of an angle. Butler and Wren suggest that construction of the various concurrent lines of a triangle will provide students with experiences which will serve to make the concept of locus meaningful.[27] They suggest also that work with concurrent lines of triangles provides the basis for construction of the circumscribed and inscribed circles of triangles, and these also are fundamental locus constructions.

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[27] Charles H. Butler and F. Lynwood Wren, *The Teaching of Secondary Mathematics*, New York, McGraw-Hill Book Co., 1941, p. 405.

The complete solution of a locus problem will always have two phases. In the first phase it is necessary to construct, and perhaps identify, a figure which satisfies specified conditions. The second phase involves proving that the constructed figure contains all points and only those points which satisfy the required conditions. An example will serve to illustrate both a reflective procedure for solving locus problems and some of the principal stumbling blocks that may be encountered. Suppose we are to solve the problem: *What is the locus of the vertex of all possible right triangles with a given hypotenuse as base?* In solving this problem, a class may well be guided into thinking along somewhat the following lines:

What is the central fact about the required locus? Is it any right triangle or is it a specific right triangle? Suppose we sketch several right triangles all having the same hypotenuse. Join the vertices of the right angles so sketched by a dotted line. Does this dotted line suggest any regular figure which might be the required locus? What is the relationship between an inscribed angle of a circle and its intercepted arc? What is true of angles inscribed in a semicircle?

By means of questions such as these, each member of a class may be led to discover for himself that the relationship of a right angle inscribed in a semicircle is the key to the problem. The final solution is then accomplished as follows:

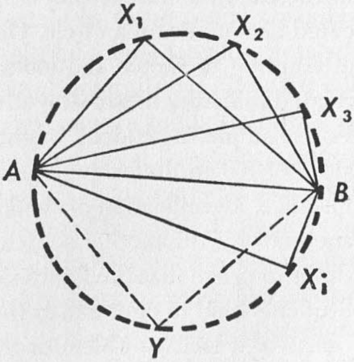

FIG. 6

PHASE I. To construct and identify the required locus.

1. Draw the segment AB representing the given hypotenuse. (See Fig. 6.)
2. Construct several right triangles with AB as hypotenuse (i.e., draw random lines AX_1, AX_2, AX_3, . . . AX_i, and drop perpendiculars from B to these random lines).

3. Draw a smooth curve through points X_1, X_2, X_3, . . . X_i, and identify the curve or figure drawn.

PHASE II. To prove that the circle on *AB* as diameter is the required locus:

a. To prove that all points which satisfy the given conditions are on the suggested locus it is necessary to prove that the vertex *X* of any right triangle *AXB* is on the circle whose diameter is *AB*. It is not difficult to show that in this instance any circle circumscribed about *AXB* will have *AB* as diameter and that, therefore, the point *X* is on the locus as drawn.
b. To prove that any point on the suggested locus satisfies the given conditions, i.e., is the vertex of a right triangle with *AB* as hypotenuse, it is only necessary to show that if segments *AY* and *BY* be drawn to any point *Y* on the circle, then the angle *AYB* is a right angle and therefore that the triangle *AYB* is a right triangle with *AB* as hypotenuse.

Hence, any point *X* which satisfies the given conditions is on the circle on *AB* as diameter and any point *Y* on the circle satisfies the given conditions. Therefore, the circle is the required locus.

Students often have trouble in identifying the locus after they have located several points which satisfy the given conditions. Thus, in the example above, they might not recognize that the points located are points on a circle. Obviously, locating more points before attempting to sketch the locus will help to avoid this difficulty. A related difficulty has to do with failure to sketch the complete locus. For example, a student might not recognize that points on both sides of *AB* in the example above satisfy the conditions set. In this instance, identification of the locus as a circle would help to prevent such error. Still another source of error is failure on the part of students to prove that the figure sketched in a problem is the true locus. Students tend to omit one or the other of the two parts of the second phase of a locus problem or even to omit the proof phase entirely.

The difficulties experienced by students in understanding and applying the concept of locus will be lessened if they gain early mastery of such fundamental concepts as the following:

1. The locus of points in a plane at a given distance from a given

point is a circle with the given point as center and the given distance as radius.

2. The locus of points in a plane at a given distance from a given line is a pair of lines parallel to the given line and at the given distance from it.

3. The locus of points in a plane equidistant from two given points is the perpendicular bisector of the line segment joining the two given points.

4. The locus of points in a plane equidistant from two intersecting lines in the plane is the pair of lines bisecting the angles formed by the given lines.

5. The locus of points in a plane equidistant from two parallel lines in the plane is a line parallel to the two given lines and midway between them.

6. The plane locus of the vertex of a right triangle with a given hypotenuse as base is a circle on the given hypotenuse as diameter.

Some of these statements of locus relationships, particularly the first two, are usually given as postulates in secondary-school geometry texts. Or they may be established through formal proofs as in the example given earlier. In either case, once these fundamental propositions are established and understood, their use will greatly facilitate solution of other locus problems and simplify necessary proofs. It will be noted that these fundamental propositions are static as stated. An excellent exercise to help clarify the two aspects of the locus concept would be to require students to restate the propositions dynamically in terms of a moving point.

GEOMETRIC CONSTRUCTIONS

Study of demonstrative geometry does not progress far before the problem of making constructions is encountered. Older textbooks in geometry usually begin with exercises involving the more fundamental constructions. With the introduction of rather extensive work in intuitive geometry in elementary grades and in the general mathematics courses at the ninth-grade level, it is not uncommon to find that pupils beginning the study of demonstrative geometry are already familiar with the mechanical techniques involved in bisecting a line segment, bisecting an angle, drawing a perpendicular to a line at a given point, drawing a perpendicular from a point to a given line, and constructing an angle equal to a given angle. Yet Minnick found that, the ability of pupils to construct figures varied widely not only between pupils of different schools but also among pupils of the same school.[28] And, as Breslich points out, despite the simplicity of these fundamental constructions, tests show that they are easily forgotten.[29] If learning is a matter of developing insights, these techniques would be less likely to be forgotten if the student had ever gained an insight into the reasons why any particular sequence of acts was performed in making a particular construction. For example, the procedure for bisecting an angle is as follows:

1. With the vertex of the angle as center and any convenient radius, strike arcs cutting both sides of the angle.
2. With any suitable radius and the points where the primary arcs cut the sides of the angle as centers, strike arcs intersecting within the angle.
3. Join the point of intersection of these secondary arcs to the vertex of the angle.

Obviously, if the process of bisecting an angle is "learned" by simply memorizing this procedure, it will be easy to forget. But if students really understand that in drawing an angle bisector they are constructing the locus of points equidistant from the sides of the angle; that the intersection of the second pair of arcs is simply one point on this locus; and that, since the vertex is also on this locus, joining the two is the obvious way of obtaining the bisector of the angle, they will not need to memorize—and remember—the specific steps required. Because they have insight into the relationships existing between the angle and the required bisector, they will be able to devise appropriate means of accomplishing the required construction. Understanding is not readily lost but *memoriter* learning of isolated facts and techniques is soon forgotten. And an individual who understands a process can readily modify it to meet altered situations whereas one dependent on *memoriter* learning is helpless in the face of changed conditions.

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[28] Minnick, *op. cit.*, p. 192.
[29] Breslich, *op. cit.*, p. 284.

A second problem involving geometric constructions has to do with construction of figures to be used in the proof of theorems and problems. If we are more concerned with the reasoning processes used in making these proofs than we are with geometric relationships, time spent in constructing accurate figures may limit unduly the time available for studying methods of proof and, particularly, for applying such methods to nongeometric situations. Obviously, after the method of constructing a perpendicular from a point to a line is once understood, students will not gain in power to do critical, reflection-level thinking by practicing this construction each time there is occasion to drop a perpendicular from a point to a line in their subsequent work in geometry. Many teachers of geometry feel strongly that all figures used should be constructed—i.e., if the figure calls for a perpendicular to be constructed, the actual construction should be carried out. Minnick believes, for example, that inaccurate and specialized drawings are fruitful sources of error in geometry.[30] Breslich likewise would permit no freehand sketching of geometric figures.[31] Yet the figure in geometry can be no more than an aid to thinking, and a student is expected to recognize that it does not represent the "real" elements he is discussing. The lines he discusses presumably have no dimensions other than length but the lines of his figure do have definite width, and so on. The Greeks, for example, restricted the instruments that could be used in constructing geometric figures to the straightedge and compass because they wanted to reduce dependence on "sensation"—on the material figures that could be drawn—and to focus attention on the generalized concepts of those figures. If a student constructs a figure with care so that it possesses certain properties, he is likely to forget that the lines and points he is discussing in his proof are not the lines and points of the figure he has just gone to so much trouble to construct. To him, the proof applies to that particular triangle, if the figure in question happens to be a triangle, and not to triangles in general. It is true that students are inclined to accept uncritically relationships that seem to exist in inaccurate or specialized figures. On the other hand, one of the pitfalls to careful thinking in nongeometric situations is just this tendency to accept uncritically tacit assumptions growing out of appearances. And, if a figure has been carefully constructed, the temptation to use apparent relationships uncritically is increased, rather than decreased.

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[30] Minnick, *op. cit.*, pp. 192 ff.

[31] Breslich, *op. cit.*, p. 262.

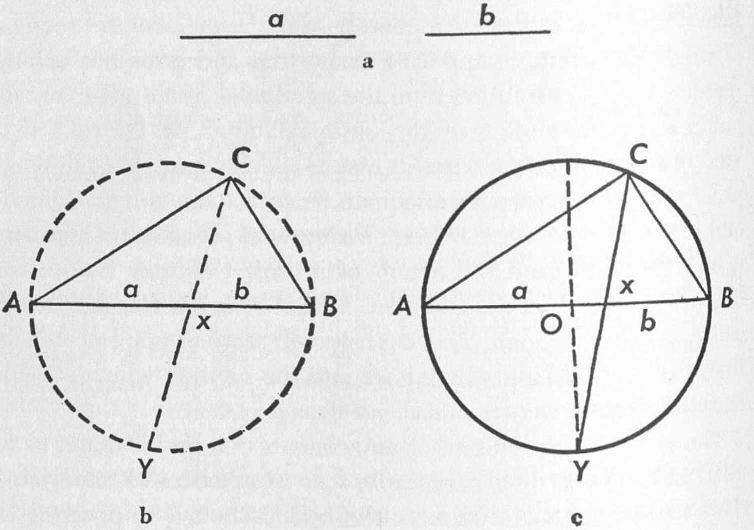
There will probably always be those who feel that acceptable standards of rigor and accuracy in teaching geometry require accurate construction of all figures used. But there seems to be little doubt that other equally earnest and competent teachers will find that the fundamental aim of developing the ability and willingness of the pupil to do reflection-level thinking can be achieved even if there is some sacrifice of rigor in the matter of constructing figures. It is never justifiable to permit students to sketch figures unless they can show how the particular line or point could be constructed with the straightedge and compass. It is equally obvious that, whether constructed or sketched, the figures used should be general rather than special and that they will be of greater value as aids in thinking through a problem if they are reasonably accurate and approximately to scale.

In the case of problems in which the question at point is the possibility of making a given construction, the same considerations hold. The educative value of a construction derives primarily from the reasoning by which the possibility of making the construction is established rather than from the actual mechanical process of drawing lines and arcs. It is possible, for example, that a student might think through the steps in a given construction problem, indicate how a solution could be obtained on a freehand sketch, prove that the suggested solution would possess the desired characteristics, and discuss the conditions under which the required construction was possible and whether or not the suggested solution was unique without ever actually making the construction with straightedge and compass. Such a solution, in so far as the fundamental aim of teaching geometry is concerned, would be entirely adequate. It would be, from the standpoint of educational value, much better than that of the student who merely stumbles on a correct sequence of ruler and compass applications by trial and error but achieves thereby a perfect solution from the standpoint of a draftsman without real understanding of the basic relationships. Of course, the kind of solution most desired is one in which adequate analysis and proof are combined with adequate performance with straightedge and compass. However, in engineering work, for example, an engineer often works out the details of a project entirely from a freehand sketch and leaves the more formal drawing to a less technically skilled draftsman. This analogy will serve to point up the fact that it is the thinking which goes into the solution which is of primary importance.

The steps in the solution of any construction problem are, as implied in the preceding paragraph, four in number: (1) analysis of the problem, (2) execution of the construction, (3) proof of the construction, and (4) discussion of the conditions under which a construction is possible. Suppose, for example, we have the problem: *Construct a right triangle having given the segments of the hypotenuse made by the bisector of the right angle.*

In this problem we have given the segments a and b (Fig. 7a). We are to construct a right triangle such that these two segments will be the segments cut off on the hypotenuse by the bisector of the right angle. A fruitful approach to any construction problem is to assume that the construction can be made and to make a sketch of the completed figure. In this case we would simply sketch a right triangle (Fig. 7b) and the bisector of the right angle. Using this sketch as an aid to their thinking, a class might then be led to attack the problem along some such line as the following:

What is the relationship of a and b to the required right triangle? What is the locus of possible positions for the vertex C of the right triangle? Evidently our problem is to locate C on the semicircle on $a + b$ as diameter so that CX bisects the right angle ACB. Suppose we complete the circle on AB as diameter and extend CX to Y. What type of angle is $\angle ACY$? How is it measured? How large is $\angle ACY$? How large is the arc AY? How could we locate Y so that the arc AY would be the required size?



a

b

c

Fig. 7

The detailed solution of the problem could then be completed as follows:

I. **Analysis:** The two segments a and b together comprise the hypotenuse. If a semicircle be constructed on the hypotenuse as diameter, then the vertex of the right angle will be on that semicircle. If we can find a point C on this semicircle such that CX is the bisector of the $\angle ACB$, then $\triangle ACB$ will be the required triangle. If CX bisects

$\angle ACB$, then $\angle ACX = \angle BCX$ and each is therefore equal to 45°. Now, an inscribed angle is measured by one-half its intercepted arc. Therefore, if we can determine a line CXY such that the arc AY is equal to 90°, the $\angle ACY$ will be 45°.

II. Construction: (See Fig. 7c.) On any random line l lay off segment AX equal to a and segment XB equal to b. On AB as diameter construct the circle O. Construct OY, the perpendicular bisector of AB. YX meets the circle at C. Join A to C and B to C. Triangle ABC is the required triangle.

III. Proof:

1. $\angle ACB = 90°$	Any angle inscribed in a semicircle is a right angle.
2. $\angle ACY = 45°$ $\angle BCY = 45°$	An inscribed angle is measured by one-half its intercepted arc and arc $AY = 90°$.
3. CXY bisects $\angle ACB$	The bisector of an angle divides it into two equal angles.

4. Therefore, segments a and b are the segments cut off on the hypotenuse of triangle ABC by the bisector of the right angle.

IV. Discussion: The construction will always be possible, whatever the lengths of a and b. Since Y might be taken at either end of the diameter perpendicular to AB, two triangles may be constructed meeting the required conditions. However, the two triangles will be congruent, so there is, in reality, only one solution.

It is at once evident that such a construction, like the sample proof given on page 155, employs both inductive and deductive reasoning. The essential thing in teaching constructions is that a student not be permitted to stop at the end of step 2. In this particular example, the fact that one angle bisector is given would seem to indicate that perhaps the key to the solution lies in construction of the inscribed circle. However, further study of the tentative sketch reveals that the key to the solution is the fact that the bisector of the right angle leads to an inscribed angle of 45°. Then the steps of the construction needed in the solution become evident.

Comparison of the steps utilized in solving the above problem with those employed in any reflective-thinking process will at once make evident the fact that these steps are exactly those of a reflective-thinking process. Hence, it is readily evident that practice in solving problems, whether construction problems as in this instance or "originals" involving various types of proofs, will be of value in giving practice in doing reflection-level thinking. In particular, if the problems arise out of experiences of the pupils—if they are problems which they recognize as problems rather than as mere school exercises—their value as a means of developing the reasoning abilities of students and of aiding them to discover opportunities for transfer of training from schoolwork to the problems of everyday living will be enhanced.

GENERALIZATION

A particularly valuable exercise from this point of view is that of generalization. After proving the various separate theorems relative to measurement of the angles formed by lines of a circle, a geometry class may profitably spend time formulating a general theorem which has all these different theorems as special cases. The reasoning skill involved here is that of formulating a general law from consideration of specific cases and is of wide applicability outside of mathematics classrooms.

SYNTHESIS OF PLANE AND SOLID GEOMETRY

An obvious area for generalization is to extend the concepts of plane geometry to relationships in three-dimensional space. Historically, solid geometry developed before plane, and the first geometric concepts a child recognizes are those of solids—spheres, cubes, cones, and cylinders, for example. Actually, the ideas of points, lines, and planes studied in plane geometry represent abstractions at a more mature level than the study of fundamental solids. Therefore, there is no real obstacle to combining the study of plane and solid geometry in one course instead of continuing the usual compartmentalization in two distinct courses. Some textbooks now available make provision for such synthesis.

If the synthesis of solid geometry with plane is to be attempted, it may be approached through generalization of the theorems of plane geometry. For example, the analogy between parallel lines and parallel planes is obvious after the concept of dihedral angles and their plane angles is developed. Less obvious on the surface but equally valuable as an opportunity for giving experience in generalization are the relationships between the congruence theorems of plane geometry and the theorems on congruence of trihedral and polyhedral angles in space. Theorems on volumes follow readily from those on plane areas. Wilt suggests that the topic of locus offers rich material for bringing together the content of plane and solid geometry.[32] The theorems of spherical geometry are particularly good as means of showing how changing assumptions as to the nature of a surface will lead to most unexpected conclusions. It seldom fails to astonish a class to learn that the sum of the angles of a spherical triangle is more than 180°, particularly if the close analogies between so many of the properties of plane and spherical triangles have been pointed out. Whether solid geometry is taught as a separate course or in conjunction with plane geometry, these contrasts and similarities should be emphasized.

GEOMETRIC FIGURES IN SOLID GEOMETRY

The matter of geometric figures as an aid in proving theorems is of particular importance in solid geometry. In the first place, a student may be disconcerted when he realizes that he cannot with exactitude show three-dimensional relationships on a two-dimensional surface. It is sometimes difficult for students to realize that an angle obviously not a right angle in the figure is, according to the hypothesis, a right angle. In fact, development of ability to visualize spatial relationships is one of the most important values to be gained from a study of solid geometry. The similarity between the problem of representing three-dimensional figures in solid geometry and problems relating to depth and perspective in drawing is at once evident.

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[32] May L. Wilt, "Teaching Plane and Solid Geometry Simultaneously," *The Teaching of Geometry*, Fifth Yearbook of the National Council of Teachers of Mathematics, New York, Bureau of Publications, Teachers College, Columbia University, 1930, p. 66.

The theorems of solid geometry do not afford any particularly valuable additional practice in reasoning beyond that afforded by a good course in plane geometry. The fundamental relationships can well be taught intuitively. Their rigorous proof is, in many cases, more satisfactorily accomplished by use of concepts from calculus and analytic geometry. But solid geometry offers unique possibilities for practice in generalization, as indicated above, and for experience in visualizing relationships which cannot be made concrete or which can be made concrete only with great difficulty. The practical applications of this ability are obvious—but should not be taken for granted by a teacher.

SUMMARY

The study of geometry, whether of two dimensions or three, can be made a valuable means of aiding secondary-school students to develop ability to do critical, reflective thinking, not only about geometric and other mathematical relationships, but about any type of problem or perplexity. However, study of geometry will not be of value in this regard unless a teacher is continuously on the alert for ways and means of giving practice in applying the concepts and techniques of reasoning learned in geometry to nongeometric situations, particularly situations growing out of pupil experiences. Any school exercise will remain just that to a student unless he becomes conscious that it touches upon a discordance or perplexity in his own life and offers a means of resolving that disharmony or perplexity. If a student discovers that the things he learns in geometry are immediately applicable to the problems and perplexities which arise in his daily living, the problem of securing transfer of training will be solved.

Properly taught, demonstrative geometry has unique value as a means of developing the ability to reason about life problems. The subject matter has value as background for work in fields as widely separated as science, art, and mechanics. It is the task of a teacher, therefore, so to conduct his classes that his pupils will gain from their study of geometry the values which are potential in the subject.

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Chapter 7

Ninth-Grade Algebra

ALGEBRA presents a sharp contrast to geometry in method and content. Geometry is primarily concerned, from the standpoint of content, with measurement. Its problems are those having to do with spatial relationships. The subject matter has been organized, since the time of Euclid, in a form which displays clearly the characteristics of a postulational science. The method of geometry is that of logical proof, of showing that a given conclusion is the necessary consequence of other propositions previously proved or accepted without proof. The educative value of geometry as a part of the secondary-school curriculum derives in large part from its characteristic method; the informational value of its subject matter is of only secondary importance. So we justify retention of geometry in the secondary-school curriculum on the ground that it can be taught so as to afford a unique and effective means of developing ability to do critical, reflection-level thinking about the problems of life.

Algebra, on the other hand, had its origin in counting and calculation and is, therefore, essentially an extension and generalization of arithmetic. Although no less logical in organization than geometry, the postulational character of the subject matter of algebra is not so readily evident as in the case of geometry and the typical method of algebra is, to an uncritical or superficial observer, a matter of manipulation of symbols rather than a process of logical reasoning. Moreover, a beginner is likely to find algebra more baffling than geometry because he finds familiar processes—the fundamental operations of arithmetic—being applied to abstract symbols in what appears to be a highly arbitrary manner. He may not understand just what is being attempted when he first encounters a geometric proof but he can readily verify the statement that line segment *AB* is equal to line segment *CD*. There may not be any good reason, so far as he can see, for wanting to know whether or not these two segments are equal but the fact of equality is something tangible—something he can verify by measurement. On the other hand, the algebraic statement that $a + b = c$ is often wholly incomprehensible to a novice. The untrained mind finds it exceedingly difficult to grasp the idea that letters such as a, b, and c may be used to represent numbers and, therefore, that c can be the sum of a and b. And, in contrast to geometry, there is no good way in which he can assure himself that the statement is valid. So, although the methods of algebra are familiar to him from his study of arithmetic, because it has a high degree of generalization and abstraction, and because it deals with processes rather than material things, the study of algebra seems intangible, confusing, and devoid of reality or utility to a beginner.

If this be true, what can be the educative value of the information about abstract symbols and the techniques of manipulating them which seems to comprise the subject matter of high-school algebra? It seems to the uninitiated that there can be little of practical value in these symbols since, to apply them to a specific problem, he must substitute numerical values for them and his problem then becomes a problem in arithmetic. Moreover, in further contrast to geometry, a beginner does not recognize any distinctive method of thinking inherent in algebraic processes that is generally applicable to nonalgebraic situations. Certainly, if algebra is merely a body of abstract symbols and a set of techniques for mechanical manipulation of these symbols and if the outcome of its study is to be mere facility in this meaningless juggling, algebra is "educational sawdust and gravel," not only for second-quarter intelligences, as Snedden charged,[1] but for any who study it.

The utility of algebra to a student who expects to take further work in mathematics or the sciences, particularly the physical sciences, is unquestioned. But such students constitute only a small proportion of the total enrollment of our secondary schools and, although it is desirable that prospective engineers, scientists, or research mathematicians enter college with a good background of mathematical skills and understandings, it is quite possible for colleges to give instruction in the concepts and techniques of secondary mathematics to those students who discover a need for such instruction after leaving secondary school.

There is little reason to question the validity of data indicating that some students lack mental ability to do satisfactory work in mathematics, particularly in formal courses in algebra and geometry. If there is definite and conclusive evidence that a student cannot profit from a study of algebra, it will of course be folly to insist that he study it. Therefore, since the needs of a student whose educational and vocational objectives are such that he must study algebra may be met at the college level, and since it is obviously foolish to insist that a dull student who cannot profit from its study take algebra, the place of algebra as a subject of instruction in the secondary school must be determined by what it can offer to those students who are neither so dull as to require special consideration nor yet specifically interested in algebra as a prerequisite to some already clearly visioned goal. In other words, is algebra of sufficient educational value, apart from its value as a prerequisite in certain technical fields, to be made an essential part of general education at the secondary level?

The proper answer to this question is a matter of bitter controversy. As Ligda says,

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[1] Quoted by William Betz in "Whither Algebra," *The Mathematics Teacher*, 23:106–107 (February, 1930).

> The amount of training in algebra actually needed by the people at large has been a burning issue in educative circles for a long time. Eminent authorities have stated that all the algebra that can be assimilated should be taught to all the people capable of learning; just as great authorities have asserted that the amount actually needed by the masses is infinitesimally small. . . .
>
> The first group of experts has so far presented but few if any valid proofs that high school algebra deserves an obligatory position in the curriculum. The members of the second group are in still worse position. Their strongest argument is founded on fallacious reasoning. They generally start from the right premise, "People use very little algebra," but arrive at the wrong conclusion, "Therefore they have very little need of it."[2]

Ligda suggests that people do not use algebra simply because they do not know how to use it:

> They are much like savages who never miss the benefits and comforts of an undreamed-of civilization. When a savage finds himself in a situation where he would be benefitted or his life saved through the use of the tools or drugs of civilization, he uses whatever makeshifts are available, prospers or fails, lives or dies, in blissful ignorance of the factors which really determine the issue. Similarly, our average citizen, when confronted by some situation requiring quantitative thinking and knowledge, calmly sidesteps the issue through the use of makeshifts, the guess and trial-and-error methods. If lucky in his guesses, he becomes a prominent business man firmly convinced of the uselessness of schooling. If unlucky, he is speedily removed from places of responsibility and his opinion is considered to be of no importance.[3]

The ultimate aim of education in a democratic society is to develop the reasoning ability of an individual to the end that he will be competent to cope with the perplexities and responsibilities of life. The really efficient person in a democratic society "is he who has somehow acquired the ability to recognize situations and quantitative relations, to gather data toward a given end, to relate cause and effect, to form correct judgments."[4] In short, the effective citizen

[2] Paul Ligda, *The Teaching of Elementary Algebra*, Boston, Houghton Mifflin Co., 1925, pp. 211–212.

[3] *Ibid.*, p. 214.

[4] *Ibid.*, p. 225.

in a democratic society is he who can, and will, do scientific, critical thinking about the problems of life as he meets them.

The methods and concepts of algebra are so much a part of scientific—reflective—thinking and writing that an individual who is ignorant of them will find the task of achieving any real understanding of the modern world difficult, if not impossible. Unquestionably, it is possible for one to live a happy and useful life without ever having understood the concepts or mastered the techniques of algebra. But a knowledge of algebra can contribute so much to competence to cope with this modern world, and its concepts and techniques are so readily applicable to problems of daily living, that no individual who aspires to be other than "a hewer of wood and a drawer of water" can afford to remain ignorant of the subject.

However, if the aim of secondary education is to develop competence to cope with the world, acquisition of factual information or technical skills or mere facility in manipulation of symbols cannot be, in themselves, satisfactory outcomes of instruction in algebra. The techniques and the symbolism of algebra together provide a most powerful method for attacking and solving problems involving counting and calculation—a method that is widely applicable because of its generality. Therefore, study of algebra will afford practice in the use of problem-solving techniques which are readily applicable to life situations. Moreover, since the subject matter of algebra represents abstractions and generalizations of specific problems, since mastery of the concepts and techniques of algebra adds to one's ability to understand and cope with the world, the study of algebra can be an important means of achieving the aims of secondary education in a democratic society. Improperly taught, algebra may be merely a pointless juggling of meaningless symbols that is worse than useless from an educative standpoint. Properly taught, it can (1) broaden and strengthen a student's mastery of arithmetic, (2) increase his power and resources for attacking and solving quantitative problems, and (3) provide a background of understanding for more advanced courses in mathematics and the sciences as well as for innumerable other aspects of modern life.

THE SUBJECT MATTER OF NINTH-GRADE ALGEBRA

The subject matter of a ninth-grade algebra course should be that material which will most effectively serve to achieve educational objectives accepted as desirable outcomes of instruction in algebra. The general topics comprising a typical ninth-grade algebra course at the present time may be classified as follows:

1. The formula as a shorthand statement of general quantitative relations. Mastery of this concept involves ability to translate into a formula any clear statement of a quantitative relation, ability to evaluate formulas of appropriate complexity, and ability to "change the subject" of a formula.
2. The equation as a statement of specific quantitative relationships and the laws governing the manipulation of symbols in the equation.
3. Extension of the number system of arithmetic to include signed and literal numbers and application of the fundamental operations of arithmetic to this extended number system.
4. Solution of linear and quadratic equations and sets of linear equations in more than one variable with applications of these techniques to practical problems.
5. Special techniques for operating with algebraic symbols such as:
   a. Special products and factoring.
   b. Operations with algebraic fractions and the solving of fractional equations.
   c. Work with exponents and radicals.
6. The concept of Cartesian coördinates and techniques of graphical representation of quantitative relationships.
7. Certain topics of considerable practical value but not essential to logical development of algebraic theory such as the concepts of elementary statistics and of numerical trigonometry.

Despite extended and bitter controversy over the value of algebra as a part of the secondary-school curriculum, the subject matter of the course has remained essentially unaltered through the years. For example, the topics suggested by Nunn differ from those listed above principally in that Nunn recommends, in addition, inclusion of work with logarithms and with fundamental concepts of differential and integral calculus.[5] In making this recommendation, however, Nunn was considering the whole secondary-school mathematics program rather than the work of the ninth grade alone and he referred to English rather than American secondary schools. Thorndike and others, in listing abilities "most worth acquiring in a year's course in algebra," present essentially the same list of topics as that now taught with the addition of work with logarithms.[6] The list suggested by Young does not include logarithms in the work of the ninth year.[7] In short, changes over the years have been changes in emphasis and in method rather than in subject matter.

The Joint Commission suggests that "By a wise and thoroughgoing elimination of unduly complicated technical processes, time has been gained in recent years for greater stress on understanding, on mastery of essentials, and on significant applications."[8] The Joint Commission attributes this development to the National Committee on Mathematical Requirements, which completed its work in 1923. In its report, this committee said:

> *The primary purposes of the teaching of mathematics should be to develop those powers of understanding and analyzing relations of quan-*

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[5] T. Percy Nunn, *The Teaching of Algebra*, New York, Longmans, Green and Co., 1914, pp. 51–52.

[6] Edward L. Thorndike, Margaret V. Cobb, Jacob S. Orleans, Percival M. Symonds, Elva Wald, and Ella Woodyard, *The Psychology of Algebra*, New York, The Macmillan Company, 1923, pp. 223–226.

[7] J. W. A. Young, *The Teaching of Mathematics in the Elementary and the Secondary School*, New York, Longmans, Green and Co., 1925, pp. 325–326.

[8] *The Place of Mathematics in Secondary Education*, the Report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, Fifteenth Yearbook of the National Council of Teachers of Mathematics, New York, Bureau of Publications, Teachers College, Columbia University, 1940, p. 87.

> *tity and space which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and action which will make these powers effective in the life of the individual.*
>
> All topics, processes, and drill in techniques which do not directly contribute to the development of the powers mentioned should be eliminated from the curriculum. . . .
>
> Continued emphasis throughout the course must be placed on the development of ability to grasp and utilize ideas, processes, and principles in the solution of concrete problems rather than the acquisition of mere facility or skill in manipulation. The excessive emphasis now commonly placed on manipulation is one of the main obstacles to intelligent progress. On the side of algebra, the ability to analyze a problem, to formulate it mathematically, and to interpret the result must be dominant aims. *Drill in algebraic manipulation should be limited to those processes and to the degree of complexity required for a thorough understanding of principles and for probable applications either in common life or in subsequent courses which a substantial proportion of the pupils will take.* It must be conceived throughout as a means to an end, not as an end in itself. Within these limits, skill in algebraic manipulation is important, and drill in this subject should be extended far enough to enable students to carry out the essential processes accurately and expeditiously. [Italics in the original.][9]

The Joint Commission suggests two alternate schemes for organizing the work in ninth-grade mathematics. One provides for a formal algebra course intended for the more able student and offers no particular innovations in either method or content.[10] The Commission suggests that there may be a place for such a course in larger schools having an appreciable number of students whose interests or vocational and educational objectives are such that a relatively formal course in algebra is desirable. The second plan proposed by the Joint Commission "presents the ninth grade program as one of general mathematics with algebra serving as the central theme."[11] Such a course, in addition to the topics in algebra listed above, would include some additional work in arithmetic and in intuitive geometry. The detailed outline of the course proposed by the Joint Commission is as follows:

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[9] The Mathematical Association of America, *The Reorganization of Mathematics in Secondary Education*, Buffalo, Mathematical Association of America, Inc., 1923, pp. 10–11.

[10] *The Place of Mathematics in Secondary Education*, pp. 107–109.

[11] *Ibid.*, p. 86.

ARITHMETIC

I. Continuing the study of arithmetical concepts, skills, and principles, primarily in connection with the work in algebra.

II. Solving applied problems, as suggested by the work in algebra; also problems involving approximate computations; the slide rule (Optional).

GEOMETRY

I. Reviewing and applying the geometric training suggested for grades 7 and 8, in connection with the work in algebra.

II. In classes of superior ability, making a beginning in the study of demonstrative geometry.

GRAPHIC REPRESENTATION

I. Reviewing and applying the techniques acquired in grades 7 and 8, in the representation of statistical data demanding greater maturity, mainly from the fields of business, economics, the social studies, and science.

II. Making graphs representing algebraic formulas such as $p = 3s$, $i = .04\,p$, and so on.

III. Using graphs in the study of linear equations.

IV. (For superior pupils) Using graphs in the study of simple quadratic equations.

ALGEBRA

I. Basic concepts.
  (1) The acquisition of the basic vocabulary.
  (2) Developing the ability to explain clearly the meaning of key concepts such as exponent, positive, negative, ratio, proportion.

II. Fundamental skills and techniques. (These should be based on an understanding of carefully considered *concepts* and *principles*. Work with polynomials should be restricted to very simple cases.)
  (1) The four fundamental operations, involving
    (a) Positive and negative numbers.
    (b) Algebraic monomials or simple polynomials.
    (c) Algebraic fractions, mainly with monomial denominators.
  (2) Special products and factoring as follows:

        - (a) Square of a binomial.
        - (b) Finding the product of the sum and difference of two terms.
        - (c) Factoring a polynomial the terms of which contain a common monomial factor.
        - (d) (Optional) Factoring trinomials of the form $x^2 + bx + c$.
        - (e) Factoring the difference of two squares.
    - (3) Powers and roots.
        - (a) Laws of exponents and their use.
        - (b) Square roots of positive numbers.
        - (c) Fundamental operations involving radicals, mainly of the monomial type.
- III. Fundamental principles.
    - (1) A study of the principles governing the fundamental operations, such as the rules of order and grouping, the rules of signs and exponents.
    - (2) A study of the principles used in the solution of equations, such as the rules of equality and of transformations.
- IV. Study of relationships and of dependence.
    - (1) By tables.
        - (a) Interpreting tables of related number pairs.
        - (b) Making tables based on formulas.
    - (2) By graphs.
        - (a) Graphs as a means of illustrating quantitative relationships.
        - (b) Making graphs based on tables of related number pairs.
        - (c) Making graphs based on algebraic expressions or formulas.
        - (d) Using graphs in the study of equations.
        - (e) Using graphs in the solution of problems.
    - (3) By formulas.
        - (a) Formulas as a means of expressing relationship or dependence.
        - (b) Making formulas based on verbal statements; on geometric figures; on tables.
        - (c) Evaluating a formula.
        - (d) Transforming a formula (only in simple cases).
    - (4) By equations.
        - (a) Equations as a means of expressing quantitative relationships.
        - (b) Solving equations of the first degree in one unknown.
        - (c) Solving pairs of equations of the first degree.

- (d) Solving fractional equations.
- (e) Solving equations of the form $ax^2 = b$.
- (f) Solving simple radical equations.
- (g) Using equations in the study of proportion and variation.
- (h) Using equations in the solution of problems stated in verbal form.

V. Using algebra in life situations and in problem-solving.
- (1) Learning to translate quantitative statements into the language of algebra.
- (2) Learning to make generalizations suggested by the techniques and principles of algebra, particularly with relation to the precise way in which definitely related, changing quantities will influence each other under given conditions.
- (3) Solving general verbal problems, using as a means of solution the table; the graph; the formula; the equation.
- (4) Applying the techniques of algebra in problem situations arising in business; in the shop; in science; in everyday life.
- (5) Interpreting the solution of equations, including negative values, where they have significance.

TRIGONOMETRY

I. Reviewing the necessary concepts and skills.

II. Finding heights or distances indirectly by scale drawing: the Pythagorean relation.

III. Finding heights, or distances, or angles, indirectly by using the natural trigonometric functions (sine, cosine, tangent).

IV. Using a table of natural functions. (Interpolation should be regarded as optional.)

THE REMAINING FIELDS

I. Mathematical modes of thinking, habits, attitudes, types of appreciation.
- (1) Continuation of the modes of thinking outlined for grades 7 and 8, as follows:
  - (a) The development of habits of correctness in computation, measurement, and drawing, and in making verbal statements.
  - (b) The development of habits of estimating and checking.
  - (c) Learning to interpret and to analyze elementary problem situations.

- (d) Learning to prepare neatly and economically arranged written solutions of suitable mathematical problems.
- (e) The development of an interest in the study of simple quantitative relationships with the aid of the table, the graph, the formula, and the equation.
- (f) Learning to understand and to apply relational thinking (the idea of dependence, of functional thinking) as a key method of dealing with quantitative changes arising in nature, in business, and in everyday life.

II. The history of mathematics.
  - (1) The story of algebraic symbolism.
  - (2) The story of indirect measurement.
  - (3) (Optional) History of signed numbers and elementary aspects of irrational numbers.

III. Correlated mathematical projects and activities.
  - (1) The place of mathematics in the modern world.
  - (2) The mathematics of business and of the shop.
  - (3) Graphic devices used in everyday life.
  - (4) Correlation with science and the social studies.
  - (5) Elementary astronomy (descriptive).
  - (6) Mathematical recreations.[12]

A comparison of topics suggested by the Joint Commission with those suggested earlier in this discussion or with the contents of almost any high-school algebra textbook will lead to the conclusion that the chief differences between the plan proposed by the Joint Commission and traditional courses in algebra are matters of emphasis, of differing degrees of complexity in the material studied, and of a more extensive use of enrichment materials from the history of mathematics and from life situations in the program proposed by the commission. As is pointed out in the Fifteenth Yearbook, "a large number of texts bearing the title *algebra* contain essentially the material that appears here under the designations *arithmetic, graphic representation,* and *trigonometry.*"[13] However, the particular title given a text or course is not important. The important consideration is that subject-matter content taught and methods used will be such that a student will grow in ability to deal with the quantitative aspects of his environment. Properly taught to students of normal ability, the content of either the more traditional type of algebra course or the course recommended by the Joint Commission will serve adequately as a means of achieving the goals set as desirable outcomes of teaching algebra in secondary schools.

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[12] Reprinted from *The Place of Mathematics in Secondary Education,* Fifteenth Yearbook of the National Council of Teachers of Mathematics, pp. 88–91. The last section, on "The Remaining Fields," is adapted from the Grade Placement Chart, pp. 246–251 of the Yearbook.

[13] *Ibid.*, p. 86.

PROBLEMS OF TEACHING METHOD IN TEACHING NINTH-GRADE ALGEBRA

The ninth-grade algebra course is primarily a matter of extending the number system of arithmetic to include signed and literal numbers. The distinctive method of algebra is use of the equation in problem solving. Logically, the place to start the study of algebra would probably be with the extension of the number system, then, after facility has been achieved in manipulating the symbols of algebra, turning to application of this symbolism to practical problems. However, the current tendency is to begin the study of secondary algebra with the equation—specifically, with the formula—rather than with extension of the number system of arithmetic. There are good arguments for beginning with either phase and, as Breslich points out, equally valid objections to either plan.[14] But as Butler and Wren have said,

> The formula provides an ideal medium for the transition from the earlier work to the more formal and systematic aspects of algebra, and a theme about which a great deal of the work of the ninth-grade course can be organized. It involves or is closely associated with a great many of the concepts of elementary algebra, the symbolic language of literal numbers, constants, and variables; the concept of dependence and function; graphic representation of relationships; substitution and evaluation; and operations with signed numbers, literal numbers, parentheses, exponents, fractions, radicals, etc. Thus it forms a core which has points of contact not

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[14] Ernst R. Breslich, *Problems in Teaching Secondary Mathematics*, Chicago, University of Chicago Press, 1931, pp. 111–117.

> only with the previous experiences of the students but with many of the topics which will be considered subsequently throughout the ninth-grade course and in later courses.[15]

Probably for reasons such as those advanced by Butler and Wren, a majority of modern secondary-school textbooks in algebra begin with a study of the formula.

Now, just what is an algebraic formula? What are the characteristics which seemingly have convinced most writers of secondary-school algebra textbooks that the formula is the logical place to start the study of algebra? How can these characteristics be utilized most effectively to develop in students the ability to think reflectively?

As we have seen (p. 185) an algebraic formula is often defined simply as "a shorthand statement of general quantitative relations." That is, algebraic formulas utilize letters and symbols to represent the words and phrases of verbal statements of rules and relationships from arithmetic and the sciences. Ninth-grade students are already familiar with the use of certain conventional symbols to represent such words as "plus," "minus," "multiply," "divide," and "equals" and with the use of letters as abbreviations for words, e.g., when the formula $A = bh$ is used as a shorthand statement for: *The area of a rectangle is equal to the product of the base times the height.* It seems reasonable, therefore, to conclude that this concept affords an excellent vehicle for introducing the extended symbolism of algebra.

But, if $A = bh$ is considered merely a shorthand statement of the rule, then it does not carry with it any hint as to how the value of b may be obtained if A and h are known. It is, in short, merely a convenient mnemonic device, not a vehicle for aiding reflection. To give work with algebraic formulas real significance as a means of developing the ability to do reflective thinking, the formula must be presented, to use Nunn's expression, "as an instrument of mathematical statement and investigation."[16] As this statement implies, there are two aspects to the study of formulas in algebra. The first has to do with analyzing relationships and formulating concise statements thereof in terms of the algebraic shorthand of the formula. The second is interpretation and use of formulas—general statements of relationships—in specific situations.

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[15] Charles H. Butler and F. Lynwood Wren, *The Teaching of Secondary Mathematics*, New York, McGraw-Hill Book Co., 1941, p. 274.

[16] Nunn, *op. cit.*, p. 63.

As an example of the first type of problem, we may take the procedure that might be used in developing a formula for expressing temperature readings in the Centigrade scale as degrees Fahrenheit. Suppose we are given a reading of 20° Centigrade and are asked to determine the corresponding Fahrenheit reading. The first step is, of course, to determine the basis for systems of marking thermometers. From their work in science, students will be familiar with the basic facts of the case: that water freezes at 32° F. and at 0° C.; it boils at 212° F. and 100° C. From these facts they can readily discover that 180 Fahrenheit degrees are equivalent to 100 Centigrade degrees and that there are, therefore, 180/100 or 9/5 times as many Fahrenheit degrees between the freezing point and the boiling point of water as there are Centigrade degrees. If the mercury has moved 20° Centigrade up from the freezing point of water, then, it must have moved 20 × 9/5 or 36° Fahrenheit up from freezing. But, since the zero point of the Fahrenheit scale is 32° below the freezing point of water, it is evident that the Fahrenheit reading of a thermometer which stands 36° F. above freezing would be 36° + 32° or 68°. Hence the reading 68° F. is equivalent to a reading of 20° C. So far, in making this analysis, only specific numbers have been used and the result obtained is a specific result. The next step, which introduces algebra, is to generalize this result. A general rule of procedure for converting Centigrade readings to Fahrenheit readings may be obtained by substituting general symbols for the specific numbers used above. It may be stated verbally somewhat as follows: The number of Fahrenheit degrees is equal to the number of Centigrade degrees multiplied by 9/5; then 32° is added to this product. If now it is agreed that the phrase "number of Fahrenheit degrees" may be represented by the symbol *F*, the phrase "number of Centigrade degrees" by the symbol C, and the usual conventions for the phrases "is equal to," "multiplied by," and "plus" or "added to" are to be used, then the rule may be written in this algebraic shorthand: $F = 9C/5 + 32$.

What, then, is the essential insight to be developed by presenting formulas as generalizations of rules of procedure or statements of relationships? Is it not recognition that the letters used in formulas stand for numbers? Thus, in the example above, the phrases "number of Fahrenheit degrees" and "number of Centigrade degrees" were used deliberately instead of the more obvious phrasing "reading in Fahrenheit degrees" and "reading in Centigrade degrees" to emphasize that the symbols F and C stand for numbers. If the letters in formulas are considered merely convenient abbreviations of key words or phrases in the verbal statement of relationships, formulas can never be anything more than aids to memory—notes or memorandums suggesting specific procedures. Too often a formula is just that to a student, a sort of nonsense rhyme for remembering a procedure that has been memorized but not understood. It is merely a machine into which he puts numbers and grinds out "answers." If it is necessary to point out that if $d = rt$, then $t = d/r$, a student does not see d, t, and r as numbers and $d = rt$ and $t = d/r$ as statements of numerical relationships. He sees the letters as abbreviations of words or phrases and the formulas as shorthand statements of rules to be followed. Mathematically, he is what chemists call a "cookbook scientist"; he merely follows formulas like recipes in a cookbook without real understanding of the relationships involved.

As part of the process of leading students to an understanding of the formula as a generalization of a verbal statement of quantitative relationships or rules of procedure, they should be given opportunity to translate formulas into verbal statements and to evaluate particular formulas for specific values of the variables. Thus, practice in solving $F = 9C/5 + 32$ for various values of C will help to develop insight into the fact that C and F stand for numbers and that the given formula is a statement of number relationships. It is not necessary always to translate a formula into the language of a specific rule. For example, $A = (B + b)h/2$ may be translated: a quantity A is equal to one-half the product of a number h multiplied by the sum of the numbers B and b, instead of always being considered as a statement of the specific relationships between the area and the bases and altitude of a trapezoid. Practice on exercises of this sort and on the evaluation of such formulas for specific numbers will help to develop facility in use of literal numbers and will go far toward establishing clear insights into the real nature of formulas and of literal numbers.

Once students see formulas as statements of number relationships, they will not have great difficulty in "changing the subject of the formula"—in solving it for some other variable. If they recognize that $F = 9C/5 + 32$ is a statement of number relationships, they will not find it hard to understand the following process:

$$\begin{aligned} F &= 9C/5 + 32 \\ F - 32 &= 9C/5 \\ 5(F - 32) &= 9C \\ 5/9(F - 32) &= C \end{aligned}$$

When this process is understood, formulas become, not merely shorthand expressions of known relationships and rules of procedure, but powerful tools for discovering new relationships and rules of procedure for novel situations.

Once the formula is thoroughly mastered, it is not difficult to introduce the idea of using literal numbers in expressing specific relationships—the equation of condition—as contrasted to the general relationships expressed by formulas. That is, the formula discussed above—$F = 9C/5 + 32$—is a general statement of the relationship between F and C; it holds for any value of either F or C. But the expression $x + 5 = 11$ is a true statement only if x is equal to 6; its validity is conditioned upon x's being 6.

Minnick suggests that "It is desirable that the pupil's concept of an equation shall involve three elements. First, the equation is a sentence; second, when an unknown is involved it is an interrogative sentence; and third, it represents an equality, or a balanced situation."[17] If a formula be presented as a declarative sentence—a statement of fact regarding number relationships—and a conditional equation, as Minnick suggests, as a question, then the essential similarity between the two concepts will at once be apparent and concepts and skills developed in working with formulas may readily be applied to conditional equations.

Use of equations in solving problems is the typical method of algebra. Obviously, mastery of the equation as a problem-solving technique should be an outcome of a secondary-school course in algebra. There are two aspects to the problem of mastering this concept. The first has to do with techniques of manipulating algebraic symbols, the second with formulation of equations leading to solution of specific problems.

Probably the most effective approach to the problem of understanding the techniques of operating with an algebraic equation is through the idea of a balance. It is not difficult to lead students to see that the equals sign implies that the quantities which make up the two members of the equation are equivalent—are of equal weight. If, therefore, any quantity be taken from or added to either side, a similar change must be made on the other side if the balance is to be maintained. In the beginning it may be helpful to use an actual set of balances with small blocks or other uniform counters to illustrate the process. Suppose the given equation is $x + 3 = 7$. If a large block is allowed to represent the x and 10 small uniform blocks are available, three of them grouped with the x block and seven on the other side, it is easy to see that taking away three blocks from the side of the unknown will necessitate removing an equal number from the other side if the balance is to be maintained. If this is done, only the unknown or x block will remain on the one side and four blocks will remain on the other. Evidently the unknown x is equivalent to four. In similar manner, the basic concepts of adding equal quantities to both sides of an equation, multiplying both members by the same number, and dividing both members by the same number may be demonstrated. In the absence of a set of balances for demonstration purposes, sketches on the blackboard will often suffice to bring out the same concept.

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[17] J. H. Minnick, *Teaching Mathematics in the Secondary Schools*, New York, Prentice-Hall, Inc., 1939, p. 119.

Closely allied to the problem of presenting algebraic symbols as numbers instead of as abbreviations is the problem of keeping the solution of equations from becoming a process of mechanically juggling symbols. If a student is allowed to "transpose" or to "cancel terms" in an equation, there is great danger that he will fail to realize that he has in reality added or subtracted the same quantities on both sides of the equation. Similarly, a student who learns to think "divide both members of the equation by the same number" or "multiply both members of the equation by the same factor" will gain a truer understanding of the operations which may be performed in solving an equation than will he who "cancels common factors" or "cross multiplies."

The implication is that the aim of instruction in algebra is development of insight into the reasoning back of algebraic symbolism and techniques. A student who has gained this insight into algebra —who understands its meaning—is able to use its symbols and techniques intelligently. A student who does not have it—no matter how facile he becomes at "transposing" or "canceling terms"—has not gained a knowledge of algebra which will serve to increase his ability to solve life problems.

How can we be sure that students who display facility in solving simple equations are not merely juggling symbols? How can we emphasize that in solving an equation we make use of a reflective reasoning process which exactly parallels the proof of a geometric theorem? Suppose the equation to be solved is:

$$2x + 5 = x - 3$$
$$x = -8.$$

Can this solution be put in syllogistic form? Is there a statement, express or implied, of a general law—a major premise? If so, what is the minor premise? The conclusion? It should be possible to lead a class to discover that the solution may readily be put in the following form:

1. Major premise: If equals be subtracted from equals, the remainders are equal.
2. Minor premise: $(2x + 5) - (x - 5) = (x - 3) - (x - 5)$.
3. Conclusion: $x = -8$.

Evidently, then, the solution of an equation is a logical-reasoning process rather than mechanical manipulation of symbols, and teaching methods should emphasize the reasoning rather than the manipulation.

It is an instructive exercise to ask a student, after he has completed the solution of an equation, to go through his solution and state why each step is possible—to "prove" his solution after the fashion of geometric proof. A few exercises of this sort will quickly reveal which students are merely working mechanically and which really understand the algebraic theory involved.

Similarly, in solving quadratic equations, students who work mechanically will betray their lack of understanding of the reasoning behind the process by errors which will be avoided by those who have gained proper insight into the theory. For example, suppose a student attempts to solve an equation as follows:

$$x^2 - 7x = 12$$
$$x(x - 7) = 12$$
$$x = 12 \quad x - 7 = 12.$$

What is the cause of his error? Evidently he has a vague idea that the solution of a quadratic is accomplished by breaking it down into a pair of linear equations. Perhaps he would be able to carry through successfully the following solution:

$$x^2 - 7x + 12 = 0$$
$$(x - 4)(x - 3) = 0$$
$$x - 4 = 0 \quad x - 3 = 0$$
$$x = 4 \quad x = 3$$

But is the key to this solution the fact that the left member is factored and each factor set equal to the right member? Or is it that, if the product of two factors is zero, then one or the other of the factors must be zero? Obviously, if the student realized that the latter statement was the key to the process of solving quadratics by factoring, he would not have made the error indicated above.

A similar failure to grasp completely the theory basic to the solution of quadratics by factoring is responsible for another common error, that made by students who attempt to solve quadratics having no common term as follows:

$$x^2 - 6x = 0$$
$$x(x - 6) = 0$$
$$x - 6 = 0$$
$$x = 6$$

Here the student has failed to see that either factor may be equal to zero and, therefore, that $x = 0$ is also a root of the equation. This error is often made when the equation is in the form

$$x^2 = 6x.$$

Similarly, solution of a system of simultaneous equations is accomplished by procedures which result in reducing the original problem to one which involves solution of a linear equation in one unknown. If emphasis in teaching is on developing this insight into the process, once this fact becomes clear to students they will not be likely to attempt solution of such problems by mechanical application of a memorized sequence of operations. They will readily learn to think through each problem, with resultant gain in ability and tendency to attack all problems reflectively.

Often a quadratic formula is used merely as a device for getting answers. If an equation has numerical coefficients, little trouble is experienced. But if one or more of the coefficients is literal, a student who has merely learned a formula and does not understand that literal numbers are *numbers* and hence may be coefficients in the sense of the formula is confused and helpless. A student who has learned to solve equations with numerical coefficients by the formula —a student who recognizes that $ax^2 + bx + c = 0$ is a quadratic but who cannot solve the equation $gt^2 + vt = s$ for t—does not understand quadratics. He has not gained insight into relationships which will enable him to attack reflectively all problems involving quadratics.

PRESENTING NEGATIVE NUMBERS

Why are negative numbers introduced in algebra? How can students be led to realize that the familiar number system of arithmetic is inadequate for their needs? If the attention of students is directed to the fact that there is a restriction in arithmetic on what subtractions may be performed but none on addition and that, despite the restriction, we habitually use ideas which actually involve violations of it, they will readily see the wisdom of seeking a means of eliminating this discrepancy in their mathematical concepts. A teacher can point out that so long as we are counting objects it is not possible, of course, to take away from a group a greater number of objects than the group originally contained. Hence, in arithmetic, which had its origin in counting, one cannot subtract a large number from a smaller. But when the concepts of arithmetic are applied to things less tangible than material objects, it is sometimes actually possible to subtract a greater number from a lesser. Thus, in playing games in which it is possible to "go set," most students will be familiar with the situation in which an individual "goes set" for more points than he has accumulated previously and whose score is, therefore, so many points "in the hole." The idea of temperatures above or below zero, even the plus and minus notation used to represent them, will be well understood by most students; and the idea of time before or after an event such as the birth of Christ is widely used and therefore familiar to students.

As soon as a class has been brought to realize that the restriction on subtraction in arithmetic is really a fundamental disharmony between theory and practice and has recognized the convenience of eliminating this restriction, the students will be ready to seek some means of resolving the conflict. Using the example of the thermometer scale as a suggestion for one means of resolving the conflict, a teacher can introduce the concept of *directed numbers* without trouble. Thus readings above zero imply movement of the mercury up from zero and readings below zero imply a downward movement of the mercury. If this concept of *directed movement* be then transferred to travel along a street or highway with some designated point as the zero point, it is relatively easy to bring a class to see the relationship between this concrete situation and the number system of algebra.

If addition of directed numbers is introduced as a process of counting with the signs of the quantities as road signs indicating direction of travel, the process of adding signed (directed) numbers is readily presented. Thus, a line may be drawn on the blackboard and taken to represent a highway with distances east of a specified point labeled positive and distances west of this point negative. It will then be easy to show that to add +3 and +4 is equivalent to counting 3 spaces to the right of zero (i.e., travel 3 miles eastward) and then 4 additional spaces (or miles) in the same direction, i.e., to a point 7 spaces to the right (7 miles to the eastward) of the starting point. Similarly, the sum −3 + (−4) can readily be shown to be −7. The sum of +3 + (−4) can be represented as travel 3 miles eastward and then 4 miles westward to give a net movement of 1 mile westward which may be represented by −1. With numerous ex-

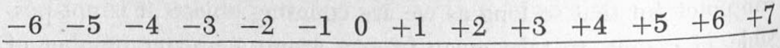

FIG. 8

amples of this sort students can be led to formulate a general rule governing addition of signed numbers.

Subtraction with signed numbers may be presented as a means of determining what must have been the initial position of an individual or object if a movement of the extent and direction indicated by the subtrahend moves him or it to the position represented by the minuend. Thus, $-6-(-2)=-4$ may be interpreted as meaning that a movement of 2 spaces to the left (2 miles westward) must have started from a point 4 units to the left of zero (4 miles westward of the zero point) if the point reached as a consequence of that movement of 2 spaces to the left is -6. As in addition, numerous examples using various combinations of minuend and subtrahend with like and differing signs will lead a class to formulate a general rule for subtraction with directed numbers.

At this point in the process, a class will have taken the first two steps in the process of solving reflectively the problem presented by the inadequacy of the number system of arithmetic in connection with subtraction. The activities of the first phase led them to recognize that a disharmony existed and to formulate the problem more definitely. In the second phase, the class gathered information about methods used in situations where subtraction of a larger number from a smaller was involved. The next step was one of generalization —formulation of a hypothesis as to the way to solve the problem. Discovery of the expedient of using directed numbers and formulation of rules for adding and subtracting them will give experience and practice in generalization. Students who have carried through these activities successfully will have demonstrated that they do have workable insights into the relationships involved.

If these basic concepts are developed by thinking reflectively and if a student clearly recognizes that it is a reflective process that is being used, algebra will not be a meaningless juggling of symbols to him. It will be what it really is, a reflective process utilizing literal symbols for unknown quantities and an example of how man may, by taking thought, invent new symbols and techniques to accomplish his ends. But unless a student actually goes through this process of thinking himself, actually participates in formulating the rules of operation, they will never be anything to him but rules to be memorized.

Rules for the operations of multiplication and division with signed numbers may be developed in similar ways. Here the problem is readily formulated: Given these new numbers, how can the remaining processes of arithmetic be extended to work with them? One of the simplest ways of presenting the ideas involved in the use of signed numbers in multiplication is that of using two scales, a time scale and a distance scale, for example. The distance scale may be taken as representing a highway with distances east or west of a specified point labeled as positive or negative. Then a time scale with some particular moment as the zero point may be set up. The plan of taking the present instant as the zero point is as simple as any for this scale. Then time past is negative and time in the future is positive. It is not difficult for students to see that a car traveling in a positive direction a given number of miles each hour will be a positive number of miles away from the starting point at the end of a given number of hours. Suppose we make the following assumptions:

1. Travel eastward is in a positive direction.
2. Travel westward is in a negative direction.
3. Time in the future is positive.
4. Time in the past is negative.

These assumptions may then be used in solving examples such as the following:

EXAMPLE 1. A car starts now and travels eastward at 50 miles per hour for 3 hours. Where is it at the end of that time?

It is evident that the rate of travel, since it is eastward, may be represented by +50. The time, since it is in the future, may be represented by +3. The problem then becomes a matter of multiplying

$$(+50) \times (+3) = +150.$$

The answer must be positive since the car will be 150 miles to the eastward of its starting point at the end of the 3 hours.

EXAMPLE 2. A car starts now and travels westward at 50 miles per hour for 3 hours. Where will it be at the end of that time?

In this case, it is evident that the rate of travel, since it is westward, may be represented by -50. The time, since it is in the future, may be represented by $+3$. The problem now becomes a matter of multiplying

$$(-50) \times (+3) = -150.$$

Here the answer is undoubtedly negative since the car traveled westward or in a negative direction and must, therefore, be in a negative or westerly direction from its starting point after 3 hours of travel.

EXAMPLE 3. A car arrives at a given point after traveling eastward at 50 miles per hour for 3 hours. From what place did it start?

Here the rate of travel, since it is eastward, would be represented by $+50$ but the time, since the car started 3 hours before the present, is -3. The problem may be written, therefore:

$$(+50) \times (-3) = -150.$$

The answer is obviously negative because, if the car could travel east or in a positive direction for 150 miles, it must have started from a point 150 miles to the westward, or from a point at a distance and direction of -150.

EXAMPLE 4. A car arrives at a given point after traveling west at 50 miles per hour for 3 hours. From what point did it start?

The rate of travel, since it is to the westward, will be represented by -50 and the time, since it was in the past, by -3. The problem is, then:

$$(-50) \times (-3) = +150.$$

The answer in this case must be positive because the car, if it traveled westward, or in a negative direction, for 3 hours before arriving at the zero point, must have started from a point 150 miles to the eastward or a positive 150 miles away.

By use of examples such as these, students may be led to formulate statements of the laws governing multiplication with signed numbers. Once they really understand multiplication with signed numbers, it will not be difficult for them to formulate comparable rules for division and for operations with fractions as well. The matter of extending the concept of signed numbers to literal numbers will not cause difficulty to students who really understand that a literal number is a number in exactly the same sense that one of the Hindu-Arabic symbols is and who also have a clear insight into the concept of signed numbers.

ALGEBRAIC POLYNOMIALS

Use of signed and literal numbers in the form of polynomials is another extension of the number system of arithmetic which is sometimes a bit difficult for beginners in algebra to understand. It is not unusual for students to understand that a letter such as t may represent a number and yet think of the expression $(t + s)$ as being simply the sum of two numbers t and s and not an individual number in itself. It is as if he saw the sum of $3 + 4$ as a sum and not as the number 7 also. Particularly in work with special products and factoring does this lack of an adequate concept of literal numbers cause trouble. As a consequence, such special products as

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)(a - b) = a^2 - b^2$$

are not recognized as relationships between arithmetic numbers. To make this fact clear to students, they should be given opportunity to use these relationships in multiplication of arithmetic numbers. For example: $23 = 20 + 3$. Therefore $23^2 = (20 + 3)^2$.

$$\begin{aligned}(20 + 3)^2 &= 20^2 + 2(20)(3) + 3^2 \\ &= 400 + 120 + 9 \\ &= 529.\end{aligned}$$

Similarly: 23×17 may be written as $(20 + 3)\ (20 - 3)$.

$$\begin{aligned}(20 + 3)(20 - 3) &= 20^2 - 3^2 \\ &= 400 - 9 \\ &= 391.\end{aligned}$$

From such exercises students will come to recognize that special products are merely generalizations of numerical relationships. From this realization it is but a short step to the insight that all algebraic expressions, be they monomials, binomials, or polynomials, are but generalized arithmetic numbers and subject to the same laws of operation. When this insight is achieved, the study of algebra will gain in interest; students will more readily discover opportunities for transfer; and their mastery of both algebra and arithmetic will be more complete and thorough.

Similarly, when factoring polynomials, if students recognize that expressions of the form $a + b$ are simply numbers, then there will not be the difficulty often experienced in problems such as

$$\begin{aligned} ax + ay + bx + by &= a(x + y) + b(x + y) \\ &= (x + y)(a + b) \end{aligned}$$

when students are puzzled as to "What happened to the second $(x + y)$?" Too often, students who would have no difficulty solving $ax + bx = x(a + b)$ are uncertain if the common factor is a binomial or a polynomial instead of a monomial. However, if they clearly understand that any polynomial is simply a number, they should not have difficulty with problems of this type.

ALGEBRAIC FRACTIONS

Operations with fractions in algebra afford an opportunity to extend the concept of algebra as generalized arithmetic still further. In the first place, such processes as finding the least common denominator in adding and subtracting fractions are not usually taught in elementary schools in a way that may be readily applied to polynomial denominators. In modern courses in arithmetic the examples studied usually involve only fractions such that the common denominators may be obtained by inspection. But in working with algebraic fractions it is necessary for students to recognize that the common denominator is the least common multiple of the various denominators. Mastery of algebraic fractions should, therefore, extend a student's mastery of arithmetic fractions—particularly if, at every step, the applications of algebraic processes to arithmetic numbers are made clear.

In particular, if students work only with algebraic fractions—if they are not led to see that they may apply the techniques developed to arithmetic fractions as well—they may develop considerable skill in manipulation of algebraic fractions and never realize that they are dealing with generalized arithmetic fractions. For example, students who have mastered only the techniques of algebra will be able to solve problems like

$$\frac{1}{a} + \frac{1}{b} = \frac{b+a}{ab}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

but those who have gained the insight that algebra is, in reality, a generalization of arithmetic will be able to recognize that the first example is merely a symbolic statement that the sum of any two unit fractions is the sum of their denominators divided by the product of the denominators and that the second example is a symbolic solution of the sum of any two fractions. Such students will have gained from their study of algebra, not only greater facility in arithmetic, but greater power in formulating general statements of principles.

EXPONENTS AND RADICALS

The topic of exponents and radicals often seems troublesome to a student because, again, he does not see expressions like $\sqrt{2}$, $\sqrt{a+b}$, and a^3 as numbers in themselves. Because he thinks of $\sqrt{2}$ as implying an operation to be performed on 2 rather than as a distinct number, he does not see that $\sqrt{2}$ may be used in any of the fundamental operations in exactly the same way that any other number may be. In fact, numbers like $\sqrt{2}$ are unknowns in exactly the same sense that the familiar x is an unknown—they are numbers whose exact numerical value is not known. Hence, if the problem of operating with such numbers is presented as simply a matter of operating with a special class of unknown numbers, it will be easier for a student to see that the laws for operating with radicals do follow the laws previously developed for working with simpler unknowns and much confusion may be avoided. Similarly, exponents may be presented as a generalized way of writing powers of numbers and the laws of exponents developed from that beginning. In particular, the relationship between radicals and fractional exponents may be readily shown by examples such as the following:

$$(\sqrt{a})(\sqrt{a}) = (\sqrt{a})^2 = a$$
$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a$$

hence, $\sqrt{a} = a^{1/2}$, and

$$(\sqrt[3]{a})(\sqrt[3]{a})(\sqrt[3]{a}) = (\sqrt[3]{a})^3 = a$$
$$a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a.$$

Whence we have $\sqrt[3]{a} = a^{1/3}$.

Again, at every step applications to arithmetic must be shown to make these algebraic manipulations have meaning. Negative and zero exponents may be made real to students in similar ways.

GRAPHIC REPRESENTATION

No discussion of the symbolic representation of quantitative ideas and relationships would be complete without consideration of the use of graphs and graphical methods. Minnick is of the opinion that "In algebra, the fundamental concept to be considered is the function in relation to the solution of real situations. The functional relation may be expressed by the equation, the formula (which is only a specialized form of the equation), and the graph. Therefore, these three should become the fundamental concepts of a course in algebra, and the mechanics included in the junior high school algebra should be those needed for the solution of problems by means of these concepts."[18] Nunn suggests that the graph shares many of the properties of the formula:

[18] *Ibid.*, p. 98.

> Like the formula it can be used to bring out and express the "law" or identity which underlies the diversity of a number of concrete numerical facts. Like the formula it delivers its message in a form readily taken in by the eye, and so ministers to the "short view." It may often be regarded as a general statement from which, as by substitution in a formula, an endless number of new particulars can be deduced. Lastly, it can in some cases be manipulated like a formula so as to yield new and unsuspected generalizations. On the other hand, it is inferior to the symbolic formula in many respects. Its accuracy depends largely on mechanical or nonintellectual conditions, such as the skill of the draughtsman and the exactness of the squared paper. It is less compact and less easily reproduced. Its message is frequently inarticulate and obscure. For these reasons it should be regarded as a *subsidiary* algebraic instrument which fulfils its best office when it either leads up to a formula by which it may itself be superseded, or serves to unfold more fully the implications of a formula whose properties have been only partially explored. [Italics in the original.][19]

In working with graphs, one must keep in mind that there are two fundamentally distinct types: statistical graphs and graphs representing functional relationships in algebra or other branches of mathematics. Statistical graphs use various schemes such as bars, segments of circles, broken lines, curved lines, and small pictures to present relationships which are comparable but not necessarily interdependent. Statistical graphs are widely used in the press and in nonmathematical fields to present data involving quantitative comparisons vividly and concisely. The essential insight desired with reference to this type of graph is ability to interpret correctly the message presented. Therefore, emphasis in teaching should be on interpretation of graphs found in daily papers, books, and magazines rather than on making graphs to present data. Some work on making statistical graphs will be necessary, of course, but ability to make a statistical graph is less important than ability to interpret one properly.

The principal function of graphs of mathematical relationships is, according to Nunn:

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[19] Nunn, *op. cit.*, p. 31.

> either to point the way to a formula by which it may itself be superseded or to unfold more fully the implications of a formula whose properties have been only partially explored. . . . The mere statement that two variables are connected by the relation $xy = a$ conveys comparatively little of its implied contents to a person who has not explored them by means of graphic analysis. . . . In this kind of application the graph not only serves to bring out particular features of the function which otherwise might not be noticed or not be realized; it also serves as a kind of challenge to the student to complete or supplement his view of the function. . . . Lastly, it must be observed that a graph is more than a means by which a given function can be conveniently explored. It is also by far the best means by which, in the earliest stages, the function is taught and symbolized. In fact, long before the student is ripe enough to discuss linear, parabolic, or other "functions" under those abstract names his mind has been prepared for the reception of the notions they connote by the repeated emergence of straight lines and parabolas in his graphic exercises.[20]

Functional graphs serve to introduce the concept of Cartesian coördinates and the idea of a one-to-one correspondence between related number pairs and the points of a plane. In teaching functional graphs, it is necessary that students be shown the correspondence between the idea of Cartesian coördinates and the scheme used in locating houses in a city, the system of United States land surveys, and the way in which geographers and navigators locate points on the earth by means of latitude and longitude. These practical applications of the fundamental notion of functional graphs will help to develop a clear insight into the nature and implications of the concept of graphical representation.

The educative uses and values of graphs, either statistical or functional, have been well summarized by Butler and Wren:

> From either point of view the graph is an effective means of presenting data, making comparisons, and depicting relations; it offers untold opportunities for free play of the imagination, for the application of simple or ingenious constructive abilities, and for the development of an enthusiastic interest in mathematical methods and a more intelligent understanding of fundamental procedures on the part of all those becoming proficient in its construction and interpretation. As the minimum contribution it

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[20] *Ibid.*, pp. 46–47.

should make to the program of attaining this proficiency, the secondary school should develop the ability (1) to construct and interpret bar, broken-line, curved-line, and circle graphs in the presentation of statistical data; (2) to make comparisons between various comparable statistical graphs; (3) to recognize the characteristics of the data to be represented by each of these four types of statistical graphs, as well as certain fundamental cautions that are to be observed in their construction and interpretation; (4) to construct and interpret a functional graph as referred to a reference frame of coordinates; (5) to use the functional graph in solving algebraic equations and to understand the simpler geometric implications; (6) to interpret the graph in the light of the functional dependence shown, including simple maximum and minimum values.[21]

SOLVING VERBAL PROBLEMS

The most important phase of algebra is the solution of verbal or "word" problems. Techniques of manipulating the symbols of algebra and for solving equations are studied for one purpose only—to give power in attacking the problems of daily living. As Schorling has said, "For what is the real purpose of teaching mathematics? Is it not to enable boys and girls to solve the problems that occur in life—in the home, in the store, in the shop? Certainly it is these problems that should be brought into the classroom whenever possible. It is important that pupils shall learn their mathematics in such a way that they will understand that their problems and their environment can be expressed in terms of it."[22] One very significant function of the verbal problem in algebra is to introduce or to show a need for new subject matter. As Thorndike said, "Other things being equal, it is better for pupils to feel some need for a procedure and some purpose in learning it before they learn it. They are then more likely to understand it and much more likely to care about learning it."[23] Sanford cites evidence to show that verbal problems for this purpose have the sanction of long usage.[24] Whenever suitable problem material is available it should be used. Thus problems about changes in temperature or about profit and loss may afford an excellent means of introducing the concept of signed numbers and for giving reality to operations with them. However, as Thorndike has pointed out, "There may be no vital, engaging problems to use as introductory material. For example, there is not, to my knowledge, any problem that is vital and engaging to the average high school pupil by which to introduce the general symbolism of fractional exponents."[25] Or a problem which might be used to show a need for a given technique may be so complicated that nothing is gained from using it to introduce the topic. And often a procedure or concept may be sufficiently interesting in itself to eliminate any need to look further for a means of motivating its study. In other words, it should be remembered that, while use of problem material is an excellent way to motivate the study of new material, it is not the only way and it need not be used if more convenient means are available.

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[21] Butler and Wren, *op. cit.*, pp. 287–288.

[22] Raleigh Schorling, *The Teaching of Mathematics*, Ann Arbor, Michigan, The Ann Arbor Press, 1936, p. 108.

[23] Thorndike and others, *op. cit.*, pp. 152–153.

[24] Vera Sanford, *A Short History of Mathematics*, Boston, Houghton Mifflin Co., 1930, p. 205.

A second use of the verbal problem in teaching algebra is closely related to the one just mentioned. It is to illustrate and to explore more fully the implications of a given concept or technique. Thus, it is not sufficient for a student to be able to solve quadratics already set up in the typical form. The computations involved are, after all, usually relatively simple. But after a student has solved verbal problems leading to quadratic equations, he will have an understanding of the nature and significance of the two roots, of the conditions which give rise to a quadratic equation, of the significance of an imaginary root, and so on, not possible without the experience given by the solution of verbal problems. Again, as Thorndike suggested, students are stimulated by seeing what a computing procedure is good for and by finding applications for the subject matter studied to problems arising in the world outside the classroom.[26]

Another use of verbal problems and problem material in teaching algebra is implied in the emphasis on problems from life outside the school. It is a means of linking algebraic theory with the practical problems of life. This may take various forms. In the one instance, a verbal problem based on practical applications of algebraic concepts and techniques may be a means of presenting useful information about significant problem situations met in daily living. Secondly, the solution of verbal problems affords opportunity for practice in doing reflection-level thinking and in using the concepts of algebra and its symbolism to express quantitative relationships of significance to the individual. Finally, use of the processes and techniques of algebra in solving practical problems not only affords opportunities for mastering these processes, as Breslich suggests,[27] but also serves to show how mathematical ideas are of assistance in interpreting the life about us.[28] Mathematical puzzles constitute still another use of verbal problems—of little practical importance, it is true, but none the less significant. As Sanford indicates, puzzle problems have long been popular.[29] Thorndike said, "Such problems make an appeal to certain human interests. Some pupils doubtless prefer them to straightforward uses of algebra in answering questions of ordinary life. The human tendency to enjoy doing what we can do well, and especially what we can do better than others can, is often stronger than the tendency to enjoy doing what we know will profit us. Some of these problems are also arranged as strong stimuli to thought for thought's sake. By introducing an element of humor, they may relieve the general tension of algebraic work as is at times desirable."[30] Thorndike was of the opinion that puzzle-type problems are more appropriate for superior students than they are for all students and that "on the whole, however, the ordinary applications of algebra to science, industry, business, and the home will give better training to the general run of high school freshmen and will inspire greater liking and respect for mathematics than will these appeals to the interest in puzzles and mystery."[31]

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[25] Thorndike and others, *op. cit.*, pp. 153–154.

[26] *Ibid.*, p. 134.

Thorndike criticized the problems in the algebra textbooks of a generation ago as being artificial and trivial. He particularly objected

[27] Breslich, *op. cit.*, p. 183.

[28] Sanford, *op. cit.*, p. 205.

[29] *Ibid.*, pp. 222 ff.

[30] Thorndike and others, *op. cit.*, p. 162.

[31] *Ibid.*

to what he called "answer known" problems, in which "in the ordinary course of events the data given to secure the answer would themselves be secured from the knowledge of the answer."[32] Such problems, according to Thorndike, were defensible only as mental gymnastics and appeals to the interest in mystery and puzzles. He said, "As such, they are better if freed from the pretense of reality. 'I am thinking of a number. Half of it plus one-third of it exceeds one-fourth of it by seven. What is the number?' is better than problems which falsely pretend to represent sane responses to real issues that life might offer. It is degrading to algebra to put it to work searching for answers which in reality would have been present as a means of framing the problem itself save frankly as a mere exercise in sharpening one's wits and in translating a paragraph into an equation."[33]

Considerable progress has been made since 1923, when Thorndike wrote the above criticisms, in providing more suitable problem material in algebra. Morever, as Breslich has pointed out, students are not greatly worried about the reality of problems they have to solve.[34] He suggests that a problem may be real and practical from an adult viewpoint and still not be suitable for elementary algebra while puzzle problems that are of little practical importance may be of real interest to students.[35] Sanford says, "In fact, even if textbooks contained no genuine problems and no recreational ones, both groups would nevertheless exist: the one because of the demands of our environment, and the other because of its appeal to our imagination. The presence of these problems in textbooks in mathematics is largely due to an opportunism which seeks to give material which men know to be of intrinsic worth or which they must admit is of inherent interest."[36] Sanford cites instances of problems which were originally included in mathematics textbooks because they represented real, practical problems but which, with the passage of time, have lost any practical content and are still retained because they serve to exemplify specific algebraic techniques better than do more practical problems.[37]

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[32] *Ibid.*, p. 138.

[33] *Ibid.*

[34] Breslich, *op. cit.*, p. 185.

[35] *Ibid.*, pp. 185–186.

[36] Sanford, *op. cit.*, p. 205.

It must be conceded that many problems found in mathematics textbooks are of types seldom encountered outside the schoolroom. As Butler and Wren suggest, "Other things being equal, problems that lie within the experiences of the pupils and that exhibit genuine reality and practicality are more desirable than those which lack those characteristics."[38] But the situations involving quantitative problems differ in school and out of school. In a schoolroom situation, the answer to a problem, even the problem itself, is a secondary consideration. The method of solution is all-important. The crucial point is whether or not the problem presents a true learning experience. Whether or not the correct answer is obtained is merely a check on the correctness of the student's thinking and an indication of the accuracy with which he uses the techniques and skills of the subject. In out-of-school situations, on the other hand, the answer is all-important and the method of obtaining it is incidental. Hence, the practicality of a problem is not necessarily a measure of its educative value. But any problem which will serve to increase a student's mastery of the techniques and concepts of algebra and, consequently, to increase his competence to solve the real problems of life outside the classroom, whether it be a real problem, a pseudo-real problem, or frankly a puzzle problem, is suitable for use in the classroom. If a problem does not stimulate growth in problem-solving ability, it is useless as a learning exercise, no matter how real it may be from a practical point of view.

Reflective thinking, the "*active, persistent, and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it, and the further conclusions to which it tends,*"[39] is the method by which problems are solved. Growth in ability to solve practical, everyday problems means, therefore, growth in ability to do reflective thinking. Dewey describes the complete process of reflective thought as follows: "Upon examination, each instance reveals, more or less clearly, five logically distinct steps: (i) a felt difficulty; (ii) its location and definition; (iii) suggestions of possible solution; (iv) development by reasoning of the bearings of the suggestion; (v) further observation and experiment leading to its acceptance or rejection; that is, the conclusion of belief or disbelief."[40]

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[37] *Ibid.*, pp. 213–226.

[38] Butler and Wren, *op. cit.*, p. 325.

[39] John Dewey, *How We Think*, Boston, D. C. Heath and Co., 1910, p. 6.

The solution of a verbal problem in algebra is a complete reflective process. The problem recognition phase (Dewey's steps i and ii) begins with reading the problem and is a matter of delimiting the problem and determining what is required for a solution. From the data given, a scheme for finding a solution is devised. This is the hypothesis. Following out this hypothesis, actually working the problem, is the fourth step suggested by Dewey and verification of the result is the fifth and final one in the reflective-thinking process.

The critical point in the solution of a verbal problem in algebra is formulation of the hypothesis—derivation of an equation, or set of equations, whose solution will give the answer sought. Generally, after this step is successfully completed, the remaining steps are reasonably simple. They are essentially mechanical applications of routine processes. The thinking is done in deriving the equation or equations. This may be regarded as a process of translating the problem from the verbal language of the book into the symbolic language of algebra. If a student is taught to be most explicit in his preliminary analysis, the thinking process is more clearly exemplified and formulation of the equation, the hypothesis for the solution, is greatly simplified. Suppose, for example, that the problem we are to solve is: *What quantity of cream testing 40 percent butter fat must be mixed with 50 gallons of milk testing 3 percent butter fat to give a mixture which is 4 percent butter fat?*

In seeking a solution, we may ask: What quantity are we required to find? What quantities do we have given? Is it the total amount of milk and cream that concerns us? Or is it the amount of butter fat?

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[40] *Ibid.*, p. 72.

Suppose we knew how much cream was to be added, what statement of equality involving this unknown could we formulate?

Our written analysis might well be about as follows:

Let $x =$ the number of gallons of cream needed.
Then we will have:

x gallons of cream testing 40 percent butter fat
50 gallons of milk testing 3 percent butter fat
$50 + x$ gallons of mixture testing 4 percent butter fat.

The key to the problem is the fact that the butter fat in the 4 percent mixture must be equal to the total amount of butter fat in its two components. Therefore, we must have:

$.40x =$ the number of gallons of butter fat in the cream
$.03(50) =$ the number of gallons of butter fat in the milk
$.04(50 + x) =$ the number of gallons of butter fat in the mixture.

Hence, we may write:

$$.40x + .03(50) = .04(50 + x),$$

which is the required equation.

It is important that students form the habit of writing out the analysis, and particularly important that it be so worded as to indicate that the literal numbers are *numbers*, not something else. Thus, in the example above, if a student were to write—as is so often done—as the first statement of his solution: "$x =$ cream added," the statement as written would actually tend to confuse his thinking. But if he specifies that x is equal to the *number* of gallons of cream, then he can more readily think of x as a number and use it in an equation.

Again, care in reading a problem will often help in making the analysis. Suppose the problem is: *One number is twice another and their sum is 84. Find the numbers.* What do we know about the two unknown numbers? Where do we obtain this information? Here the solution is implicit in two statements. From the statement *one number is twice another*, we obtain the following:

Let $x =$ the smaller of the two numbers.
Then $2x =$ the larger of the two numbers.

From the statement *their sum is 84,* we may write:

$$x + 2x = 84.$$

Solution readily follows. Students will often want to omit the written analysis on problems as simple as this. But insistence by the teacher on a detailed and accurate analysis of each problem solved will aid in development of orderly habits of thinking in attacking problems and give facility in solving more difficult problems later. The aim of these simple problems is not to find the answer—we do not particularly care what the two numbers are whose sum is 84—but to develop habits of thought appropriate to the solution of all verbal problems. Hence, insistence on written analysis of all verbal problems is but one means of insuring that students actually carry through all the steps of reflective thinking in each instance.

One of the errors commonly made in teaching algebra is to teach the solution of verbal problems by types (i.e., "work" problems, time, rate, distance problems, "mixture" problems, etc.). Thus some textbooks will group together a series of problems of the type *If A can do a piece of work in 4 days and B can do it in 6 days, how long will it take both working together to finish the task?* with suggestions as to the technique of solving that particular type of problem.

Students often memorize the particular trick involved—in this instance the use of reciprocals—often without ever seeing the reasoning behind it. Problems such as the example just given derive their educative value from the practice in analysis which led to the use of reciprocals and not from the practice in computation which they afford. It is neither possible nor desirable to attempt to classify all the types of problems which might be encountered and then to teach appropriate techniques for the solution of each. Such a practice would negate the very reason for studying algebra at the secondary level. Problems should not be taught as "work" problems, "mixture" problems, or "time, rate, distance" problems, but each type should be used to illustrate how to reason from the data given to reach the required solution and to give practice in formulating equations expressing the given relationships symbolically.

It is generally agreed that students find the solution of verbal problems one of the most difficult parts of algebra. The difficulties encountered do not ordinarily stem from the algebraic processes employed since the verbal problems found in algebra textbooks usually present relationships which may be expressed by means of relatively simple equations. Most typically, students have trouble in translating the verbal language of the problem into algebraic symbolism, partly because they have difficulty in reading the problem material with understanding. In the first place, there is a technical vocabulary to be mastered. This mathematical language includes (1) number symbols and literal numbers, (2) symbols indicating operations, (3) methods of showing relationships, and (4) technical terms which may be words in common use but which have a specialized meaning in mathematics. Such words as *right, normal, rational, radical, even,* and *odd* have specialized meanings in mathematics that are far removed from the meanings they have in ordinary usage. Secondly, mathematical subject matter requires slow, intensive reading and the student who has been taught to do rapid, cursory reading in other fields must learn new techniques when he comes to read it. Butler and Wren are correct in saying that "the ability to read analytically requires patience as well as concentrated and sustained attention. These characteristics can generally be developed to a satisfactory degree only by special training giving conscious attention to them. *The teacher of algebra must assume responsibility for giving this special training in careful analytical reading if he expects to have his students become proficient in solving verbal problems.*" (Italics mine.)[41]

A second source of difficulty with verbal problems stems from the fact that students often do not know some of the related facts which must be used to solve a given problem. Frequently some fact from science or some relationship between units of measure is essential to the solution of a problem and is not given in the data of the problem. For example, problems dealing with the law of levers may cause trouble for students, not because of any algebraic difficulty, but because they do not know the law of levers. Problems having to do with the values of digits in numbers, problems that are often found in lists of verbal problems illustrating the uses of systems of simultaneous linear equations, may be difficult to the student because he does not realize that, if he lets x be the tens digit and y be the units digit, the value of the number must be $10x + y$ rather than xy.

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[41] Butler and Wren, *op. cit.*, p. 318.

Still another source of difficulty with verbal problems grows out of the fact that a student may not be giving his work the sustained attention necessary to carry out analysis of the data and to arrive at a solution of the problem. As Butler and Wren suggest, "The careful analysis of problems requires much patience, concentrated attention, and the willingness to take the time to write down and organize all relevant data with painstaking care. These are not generally to be regarded as normal characteristics of normal, healthy young children."[42] The sustained mental effort required to carry through the reflective-thinking process necessary to solve a verbal problem in algebra often needs more maturity of purpose than many students at the ninth-grade level possess. But growth in this regard should be one of the outcomes of instruction in algebra, and particularly of the work with verbal problems.

SUMMARY

If the thinking process rather than computation or symbolism is the point of focus in teaching algebra, its study will contribute to the aim of developing each student toward becoming an independent learner competent to assume the responsibilities of citizenship in a democratic society. Because he is conscious of the reasoning processes which underlie the techniques of algebra—because he recognizes them as generalizations of arithmetic techniques—a student who really comes to understand algebra will gain increased mastery of arithmetic from his study of algebra, will gain added facility in solving the problems arising out of the quantitative aspects of his daily life, and will gain in power and disposition to think reflectively.

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[42] *Ibid.*, p. 319.

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Chapter 8

Ninth-Grade General Mathematics

THE typical secondary-school course in general mathematics at the ninth-grade level differs markedly in aim as well as in content from the more traditional algebra and geometry courses. It is, according to Schorling, "an introductory, basic, exploratory course in which the simple and significant principles of arithmetic, algebra, intuitive geometry, statistics, and numerical trigonometry are taught so as to emphasize their natural and numerous interrelations."[1] As this definition indicates, it is a composite course cutting across traditional subject-matter boundaries and including some material not ordinarily considered at the secondary level. It is expected to be a terminal course rather than one preliminary to more advanced work in mathematics. It is designed to meet more adequately the needs of that large group of students who display little or no aptitude or liking for mathematics, whose attitude toward any course in that field ranges from bored sufferance to active and intense dislike, and whose educational and vocational plans do not include serious study of the sciences or of any of the precise disciplines which require an extensive knowledge of mathematics.

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[1] Raleigh Schorling, *The Teaching of Mathematics*, Ann Arbor, Michigan, The Ann Arbor Press, 1936, p. 52.

Historically, the first proposals for modifying the traditional sequence of secondary mathematics came from mathematicians, notably Professor E. H. Moore of the University of Chicago, and were aimed at improving instruction in mathematics. In an address delivered before the American Mathematical Society at its ninth annual meeting, December 29, 1902, Moore pleaded for closer coördination of pure and applied mathematics and proposed that both the methods and the content of the mathematics taught in elementary and secondary schools be enriched and made more vital and effective. In particular, he suggested that the algebra, geometry, and physics taught in secondary schools be organized "into a thoroughly coherent four years' course comparable in strength and closeness of structure to the four years' course in Latin."[2]

The typical course in general mathematics which is taught in American secondary schools today differs in important respects from that envisioned by Moore. He evidently had in mind what Schorling terms a "correlated" course in which the elements of the traditional subjects were to be fused into a single coherent course intended to give more effective preparation for college mathematics. The modern course is definitely not college preparatory in aim and content. The subject matter represents a selection from the elements of the traditional mathematics subjects rather than an attempt at fusion and correlation. Moreover, the modern course is only one year instead of four years in length.

An attempt was made by members of the mathematics faculty of the University of Chicago High School to develop a correlated course such as Moore advocated. George W. Myers, Ernst R. Breslich, and others experimented with various plans for interweaving the subject matter of algebra and geometry. A series of textbooks incorporating their ideas was prepared. Continued experimentation, research, and revision of materials over the years by the staff of the University High School has resulted in development of a mathematics program at Chicago covering the entire secondary-school period in which there is the interweaving and fusion of traditional subject matter advocated by Moore.[3]

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[2] *Ibid.*, p. 43.

However, as a consequence of the far-reaching social changes of the first half of the present century, the high schools of America are crowded with students whose needs, interests, and vocational plans are not served by college preparatory mathematics courses. School administrators charged with the responsibility of providing a suitable program of secondary education for these students have long recognized that there was need for a mathematics course which would (1) provide an opportunity to obtain a more complete mastery of the fundamental concepts of mathematics by non-college-preparatory students; (2) provide sufficient exploratory work in mathematics to enable the student who was still uncertain about his future vocational and educational plans to make an intelligent decision as to what further study of mathematics, if any, he should make; and (3) provide the likelihood of being taught successfully to the less able students. The subject matter of this course, experience indicated, should afford students (1) opportunity to acquire greater facility in the use of arithmetic—particularly in operations with common fractions, decimals, and percents; (2) sufficient acquaintance with the basic concepts of algebra and geometry to carry on the usual activities of business and industry; and (3) acquaintance with the elementary concepts of courses such as statistics and trigonometry.

But selection of suitable subject matter would be only part of the problem. Because the students enrolled in such a course would be those whose cultural backgrounds, academic abilities, and interest in mathematics were likely to be somewhat limited, the teacher of the course would need to be particularly skilled in working with slow-learning students and in creating and maintaining attitudes favorable to successful study of mathematics.

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[3] For a description of their work, see *Mathematics Instruction in the University High School*, Publications of the Laboratory Schools of the University of Chicago No. 8, Chicago, University of Chicago Press, November, 1940, pp. 1–12. By members of the Department of Mathematics of the University High School.

THE PROBLEM OF PUPIL ATTITUDE

Even the most painstaking and skillful work on the part of a teacher will probably be fruitless unless his students want to learn. Therefore, a favorable attitude toward a subject on the part of the learner is particularly important in the successful teaching of that subject. A most striking factor in any analysis of the problems of mathematics education is the widespread antipathy and fear toward all forms of mathematics which characterize the attitudes of students and adults alike. Is it not likely that this fear or dread of mathematics derives, at least in part, from the fact that many persons have become convinced (1) that mathematics is so difficult that only a select few may hope to master it and (2) that mathematics, as it is usually taught in school, has little or no practical value to an average person? Is this the reason that people who would instantly resent any imputation that their mental abilities were below average will freely admit, even boast of, utter inability to master the simplest mathematical concepts? Yet individuals who profess the utmost ineptitude in all things mathematical will, in their conduct of practical affairs, often demonstrate mathematical ability and understanding of high order. Moreover, this antagonistic and defeatist attitude toward mathematics seems definitely to be a learned reaction since small children like to count and to play number games. Indeed, if interest in trick problems and mathematical puzzles is any criterion, adults also enjoy playing with mathematical ideas. But for some reason, whether it be uninspired teaching or something else, many persons have come to feel so inadequate in mathematics that they take refuge in pretended incompetence and an apparent disinterest in all things mathematical.

How can a teacher of mathematics overcome this dread and fear of the subject on the part of students which makes his teaching problem so much more difficult? What can he do to create a more desirable attitude toward and a more intelligent interest in mathematics?

It is frequently said that problems must be real to hold a pupil's interest. What is a real problem? The trick problems and puzzles which interest even those who profess the greatest disinterest in mathematics are certainly not real problems in any practical sense of the word. To be real to a student, a problem must offer him a challenge that he feels able to meet; it must pose a perplexity that he wants to resolve; and it must involve relationships that are within the range of his experience. Many a problem taken from real-life situations has no appeal to an adolescent boy or girl, because he or she is not familiar with the situation in which the problem arose or is not concerned with the matter with which the problem deals. The same boy or girl may be greatly interested in and challenged by a purely academic, puzzle-type problem. Any problem, whatever its source or nature, that stimulates an individual to initiate and carry through to its conclusion a reflective process is real to that individual. The problems studied in mathematics classes are, too often, problems which do not present any challenge to students because they have to do with adult concerns. If a teacher can use mathematics as a means of leading students to discover and resolve inadequacies and disharmonies in their own outlooks, can present the facts and techniques of mathematics as means by which a student can make his own world more intelligible, then that student will soon come to see the significance and worth-whileness of his work in mathematics. And if a teacher is careful to arrange the teaching situation so that an individual is never asked to do work beyond his capacity, if the procedure is varied sufficiently to maintain interest, and if care is taken that practice for mastery of skills and techniques —drill—does not become monotonous drudgery, the hampering dislike and dread of mathematics will be largely overcome.

THE PROBLEM OF THE SLOW-LEARNING PUPIL

What is the criterion by which we judge one pupil to be "slow," another to be "normal," and a third to be "gifted"? Is not the difference between them essentially a matter of learning speed? Is not the common measure of academic ability—the I.Q.—essentially a measure of the relative speed at which an individual learns? Can we say, then, that a "slow" student is one who learns more slowly than an "average" or "normal" student?

It seems probable that there is no essential difference in the way slow students learn as compared with those more gifted. The difference is in the rate at which learning progresses. A slow student grasps the relationships among the factors of a situation less readily; it takes him longer to discover the full meaning of a concept. In gaining insights into relationships, the slow student progresses by shorter steps than do his more gifted classmates. It is always a mistake, therefore, to hurry the slow pupil. There is a reasonably high correlation between speed and accuracy in computation, and usually, but not always, the pupil who works rapidly really understands what he is doing. But sometimes the slower pupil also understands—or would if he were given time. Undue emphasis on speed in computation is definitely undesirable if a teacher is working with slow pupils.

Similarly, in working with slow-learning students it is a mistake to assume that they have mastered thoroughly the work of previous grades or to slight the fundamentals of a subject because the teacher gets bored with the elementary parts and is in a hurry to get to the more advanced and difficult concepts which offer more interest and challenge—to the teacher. Often, a contributing factor to a slow-learning student's ineptitude in mathematics is his inadequate mastery of fundamental concepts that he might have grasped had the pace of the class been slower. Because of the sequential nature of mathematical subject matter, the effect of failure to grasp key concepts upon ability to master subsequent ideas is cumulative. A wise teacher will give his slow-learning students ample opportunity to discover and make up any deficiencies in their knowledge of the basic concepts of mathematics.

The slow pupil, no less than the bright, is thrilled when he achieves something or discovers a new relationship. We are all continually seeking to discover new relationships in the world about us and reorganizing our concepts about things in consequence. Other things being equal, a teaching device which causes a pupil to do something will be more successful in leading him to make these discoveries of new relationships than one which merely requires him to sit and listen or observe. Actually performing an operation helps a student to see the relationships involved, and the confidence gained from successful effort creates interest in an activity and the desire to repeat it. Again, an enthusiastic teacher will impart interest by contagion; and the minor successes achieved under the impetus of this vicarious enthusiasm will spur pupils to further efforts.

If a teacher is careful to provide a variety of activities calculated to aid students to discover relationships and to gain new insights, if he avoids overlong and overdifficult assignments, and if he is careful not to permit his pupils to become confused, he will be able to create and maintain as much interest and enthusiasm in a general-mathematics class of slow-learning pupils as in an algebra or geometry class of bright students. And, in classes in which less able students predominate, if the rate of progress is slowed somewhat below that which more gifted pupils could maintain, if the subject matter presented is carefully selected with the needs and limitations of these slow-learning students in mind, and if it is so organized that the steps in learning complex skills are suited to the capacities of the learners, slow-learning students may come to enjoy the study of mathematics and to make reasonably satisfactory progress in it.

GEOMETRY IN THE GENERAL MATHEMATICS COURSE

As has been mentioned, the general mathematics course will contain subject matter taken from arithmetic, algebra, geometry, trigonometry, statistics, and related fields. The geometry traditionally associated with the secondary school is, of course, demonstrative geometry, in which the emphasis is on method of proof rather than informational content. The geometry included in a course in general mathematics is usually what is known as informal or intuitive geometry. In contrast to the formal proofs of demonstrative geometry, in intuitive geometry the relationships between points, lines, and planes are established by measurement, experiment, and inference. The geometrical part of the general mathematics course is intended to extend and systematize the somewhat limited knowledge of geometrical principles acquired in the elementary grades and to provide an informal treatment of the geometric ideas which are an essential part of the education of all persons.

According to the Report of the National Committee on Mathematical Requirements:

> The work in intuitive geometry should make the pupil familiar with the elementary ideas concerning geometric forms in the plane and in space with respect to shape, size, and position. Much opportunity should be provided for exercising space perceptions and intuition. The simpler geometric ideas and relations in the plane may properly be extended to three dimensions. The work should, moreover, be carefully planned so as to bring out geometric relations and connections. Before the end of this intuitive work the pupil should have definitely begun to make inferences and to draw valid conclusions from the relations discovered. In other words, this informal work in geometry should be so organized as to make it a gradual approach to, and to provide a foundation for, the subsequent work in demonstrative geometry.[4]

As is evident from the quotation, the National Committee considered the work in informal geometry to be preliminary to the more formal study of demonstrative geometry. Breslich likewise considered informal geometry valuable as a means of preparing students for study of demonstrative geometry. He suggested that, if students have not had instruction in informal geometry before they reach the tenth grade, they should then have introductory work of that type before beginning their study of formal, logical, demonstrative geometry.[5] Apparently Breslich believed that demonstrative geometry with its emphasis on methods of proof is not as well adapted to presentation of elementary ideas about geometric forms and relationships as is informal geometry. He was of the opinion, however, that the best place for a study of intuitive geometry is in the seventh and eighth grades because "The simple geometric facts which are not new and therefore of little interest to the tenth grade pupil still appeal to the seventh grade pupil as interesting and worthwhile material."[6]

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[4] National Committee on Mathematical Requirements, *The Reorganization of Mathematics in Secondary Education*, Boston, Houghton Mifflin Co., 1923, p. 32.

[5] Ernst R. Breslich, *Problems in Teaching Secondary School Mathematics*, Chicago, University of Chicago Press, 1931, p. 234.

However, although intuitive geometry is accepted as an integral part of the general mathematics course as taught at the present time, instead of serving as preparation for further work in geometry it usually constitutes a student's terminal contact with instruction in the subject. Work in geometry in a ninth-grade general mathematics course should be planned, therefore, to impart information, skills, and understandings which are applicable to problems of daily living rather than techniques and concepts which are primarily of value in further study of mathematics.

What knowledge of geometric facts and processes is most readily applicable to problems of everyday living? Problems grow out of inadequacies or conflicts in an individual's outlook—his knowledge and understanding of his world. Resolution of these inadequacies and disharmonies—solution of his problems—can take place only after he recognizes that a problem exists and becomes concerned about finding a solution. The function of subject matter in education is to aid students in discovering and resolving conflicts and inadequacies in their outlooks—to aid them in arriving at dependable insights into the relationships of their world. The geometric subject matter included in a general mathematics course should be those facts and concepts about spatial and quantitative relationships which are essential to accurate knowledge of the world and which will stimulate students to think reflectively about that world.

Analysis of textbooks and courses of study indicates that the geometric materials in a general mathematics course usually include:

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[6] *Ibid.*, p. 235.

1. Fundamental ideas concerning geometric forms and relationships in a plane and in three-dimensional space.
2. Measurement of line segments and angles and acquaintance with various conventions of terminology and notation commonly used to express relationships involving these forms.
3. Distinctive properties of such plane figures as the triangle, various quadrilaterals, the circle, and common solids.
4. Computation of areas and perimeters of plane figures, the circumference and area of circles, and the surface areas and volumes of the common solids such as triangular and rectangular prisms and pyramids; of cylinders, cones, and spheres.
5. Special forms and relationships such as the angles of parallel lines; isosceles, equilateral, and right triangles; squares, rectangles, parallelograms, and trapezoids; and the Pythagorean relationship.
6. Concepts of congruence and similarity, including the basic concepts of simple numerical trigonometry, the use of scale drawings, and other applications.
7. Fundamental constructions with ruler and compass—to bisect a line segment; to construct a perpendicular to a line either at a point on the line or from a point not on the line; to bisect an angle; and to construct an angle equal to a given angle.
8. Such relationships between circles and lines in a plane as are implied by the terms *concentric, tangent internally* and *tangent externally, chord, secant, central angle,* and *inscribed angle.*
9. Development of an extensive vocabulary of geometric terms.
10. Basic principles of ratio and proportion.
11. Use of the formula as a means of expressing geometric relationships.

But study of geometric subject matter, even mastery of it in the sense of ability to state relationships or to solve school exercises correctly, does not insure growth in competence to cope with the world—to solve the problems and resolve the conflicts which are discovered in the course of daily living. The values to be gained from study of particular subject matter are not inherent in the material studied; they are potentialities which may be realized from it if properly used. It is the task of a teacher, therefore, to devise methods and procedures which will stimulate his students to make effective use of the facts and techniques of geometry in solving life problems. The teacher of general mathematics classes must use the subject matter of informal geometry to give his students opportunities to do reflective thinking and to gain added insights into the nature and purpose of reflective thinking.

Slow-learning students and students whose interest in and aptitude for mathematics is low are likely to predominate in general mathematics classes. Such students do not readily comprehend abstract ideas, hence will develop desired insights into geometric relationships and gain in power to use them in solving problems more readily from experiences with concrete objects. Therefore, it is advisable to encourage as much physical activity as possible in teaching informal geometry to classes of these less able students. For example, the preliminary unit may very well deal with measurement of line segments and angles. It may incorporate work with both English and metric units, the use of squared paper in measuring and comparing lengths and areas, and the use of the compass in transferring or comparing line segments. Use of a protractor in measuring angles and in constructing angles equal to given angles would be an important part of the work of this unit. The activities carried on by students may well include making scale drawings of classrooms, athletic fields, and other convenient areas. Actual field measurements to obtain data for such drawings afford additional experience in measuring line segments. Construction of graphs, particularly bar graphs, also affords opportunities for measurement of lines and for computations involving lengths and proportions.

Similarly, work with scale diagrams affords an excellent opportunity to call attention to the need for and convenience of a consistent system of labeling points, lines, and angles. The usual convention of using capital letters for designating points and small letters for lines, together with the various ways of designating angles, may be introduced at this time.

Presenting the idea of an angle both as a difference in direction between two lines (the static concept of angle) and as the amount of turning of a line about a point (the dynamic concept) is only one of the many opportunities which an alert teacher will discover for presenting geometric concepts as ideas in conflict and thereby stimulating a reflective study leading to wider and more harmonious insights into familiar relationships and to heightened abilities to resolve other conflicts as they are discovered. At the same time, students can readily be brought to recognize that mastery of the technical vocabulary of terms used in discussing various types of angles, particularly the more common ones such as *acute, right, straight, adjacent, vertical, complementary,* and *supplementary,* will add greatly to the facility and precision with which they can use these concepts. From this realization, in turn, will come added awareness of the value of precise formulation of ideas. Again, many concepts involving angles, such as *base angles, interior* and *exterior angles,* and *opposite angles,* may well lead to or be studied in connection with work with polygons.

The fundamental constructions with ruler and compass afford many opportunities for manual activities by pupils and for creative work in developing original designs in which use is made of these constructions. These basic constructions also afford opportunities to develop such relationships as conditions of parallelism and locus relations of angle and segment bisectors and of concurrent lines of triangles. Measurement of segments and of angles will often lead to discovery of new relationships—new to the students, that is—and formulating general statements of the relationships so discovered will give valuable practice in generalizing and in discovering properties by inference.

Study of the properties of congruence and similarity in polygons will also afford practice in mensuration, in computation, and in application of the construction techniques previously studied. Applications to practical problems should be stressed and particular attention paid to special cases such as isosceles and equilateral triangles, 30°–60° and isosceles right triangles, squares, rectangles, and parallelograms.

Study of the circle and its related lines should include such concepts as relationships between radii, diameter, circumference, and tangents. The concepts of central angles and inscribed angles should be developed and the possible relations which two circles in a plane may have for each other—together with practical applications of each idea. The concept of π as a ratio and ability to use the formulas for finding area and circumference of a circle in practical computations are still other examples of ideas and skills which help to make the world about us more intelligible.

Again, work with solids will provide experience in recognition of simple solids and knowledge of their distinguishing characteristics. Problems about surface areas and volumes will afford practice in computation and in working with formulas. Exercises in sketching the simpler solids in various positions will help to develop ability to visualize spatial relations. The problems growing out of the difficulties encountered in drawing figures of solids showing various relationships are both fruitful sources of ideas in conflict and means of emphasizing the importance of perspective and other basic ideas used by artists and draftsmen in the pictorial representation of objects in space.

Since the aim of any classroom procedure is to help a pupil progress along the road to becoming an independent learner able and willing to think for himself, emphasis in all work in intuitive geometry should be on understanding rather than on facility in computation or mechanical skills in performance. As the work proceeds, a student should learn to generalize the relationships discovered by measurement and experiment—by intuition—and to derive other relationships by inference from those already established. In short, he should progress from intuition toward reflective reasoning as a means of discovering and verifying relationships—of acquiring information and gaining insights which add to his competence to cope with his world.

ALGEBRA IN GENERAL MATHEMATICS

Algebra serves as the central theme of the ninth-grade course in general mathematics as taught at the present time. And, as is the case with intuitive geometry, work with algebra in general mathematics will probably be the terminal experience with instruction in that branch of mathematics for many students. Such being the case, the algebraic concepts and techniques presented should be those which will most effectively serve to enhance a student's ability to understand and to cope with the world about him rather than those which are primarily preparatory to further study of mathematics.

Algebra is, essentially, a generalization and extension of the number system and symbolism of arithmetic. Through use of algebraic symbols and techniques a student is able to express quantitative concepts and relationships with greater facility and precision than is possible in the more restricted notation and symbolism of arithmetic. This greater facility and precision in dealing with such ideas made possible through use of algebra will effectively increase a student's ability to understand and deal with quantitative factors in the world about him. Therefore, the work in algebra included in general mathematics should provide opportunity for less able or nonmathematically inclined students to discover and become familiar with the potentialities and utility of algebraic concepts.

Since students will be familiar with the idea of a formula from their work in arithmetic and since formulas enter extensively into statements of relationships studied in intuitive geometry, work with the algebraic concept of a formula may well be made the beginning unit of a year's work in general mathematics.

The fundamental new idea in the concept of the formula is use of literal numbers. The simplest approach is, of course, to present formulas as shorthand abbreviations of arithmetic rules. Thus, a student will be familiar with the relationship that the area of a rectangle is equal to the product of its length by its width. He has used this rule and seen it written both in the form

$$\text{Area} = \text{length times width}$$

and as a formula:

$$A = lw.$$

But, so presented, formulas are merely devices for stating and remembering rules. How can the concept of an algebraic formula be presented to students so that emphasis will be on its unique character as a generalized statement of number relationships and its utility as a means of increasing their competence to cope with quantitative aspects of their world?

If this end is to be achieved in working with formulas in general mathematics classes, the crucial concept to be mastered is that literal numbers are *numbers* and not shorthand symbols or abbreviations for words. A teacher must emphasize that A in the formula mentioned above stands for the *number* of square units in the area of the rectangle. It is not an abbreviation of the word *area.* Similarly, l denotes the number of units in the length and w the number of units in the width of the rectangle. So considered, "changing the subject of the formula"—expressing the width, for example, in terms of length and area—is merely a matter of dividing both members of an equation by the same number, not one of writing a new rule. In this connection, if much practice is given in substituting numerical values in formulas, it will help to emphasize that the letters of formulas do stand for numbers. Young says, "The numerical evaluation of algebraic expressions by the substitution of specific numbers for the letters cannot readily be overdone in beginning algebra."[7] Once a student grasps the idea that a formula is essentially a statement of number relationships, it may be used to introduce the idea of symbolic representation. Attention may be called to the fact that π, a symbol with which he is familiar, is a specific number and that this literal symbol is a convenient way of expressing this number, whose value is not definitely known. Similarly, the use of symbols such as +, =, and ÷ facilitates writing statements of number relationships. Practice in expressing known number relationships by formulas will help pupils to realize that letters may usefully represent numbers, and additional practice in substitution of numerical values in the formulas so written will help to make clear that a formula is a generalized statement of number relationships.

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[7] J. W. A. Young, *The Teaching of Mathematics in the Elementary and the Secondary School,* New York, Longmans, Green and Co., 1925, p. 298.

After the idea that letters may stand for numbers is well established, the process of solving a formula for specified letters—"changing the subject of the formula"—may be utilized to introduce a study of the laws governing solution of equations. Thus, after a student has written the formula

$$p = 2(l + w)$$

and learned to recognize it as a statement of a relationship between three numbers p, l, and w rather than as a shorthand statement of the relationship between the perimeter and the dimensions of a rectangle, the problem of setting up a similar formula for the width of a rectangle in terms of its perimeter and its length will afford an opportunity to present the axioms of addition, subtraction, multiplication, and division as applied to equations in a situation in which students can readily verify the reasonableness of the results obtained. If these axioms are applied to equations where this meaningful interpretation is not readily available, these processes are likely to become mechanical. Young says:

> All the operations of elementary algebra should be thought of as performed on numbers, not on the literal symbols for the numbers. After a time and gradually this thought may fall into the background, when the character of the symbolism and of the operations with the symbols has been thoroughly grasped. But there should never be any difficulty to pass from the symbol to the thing signified (and for quite a while at least the meaning of the symbol should be kept constantly in the foreground). This may be achieved by continually replacing the letters which represent numbers by actual numbers. Teachers of physics often find that pupils see no numerical meaning in the formulas of algebra, that they are not

able readily and properly to interpret and apply the formulas in numerical instances.[8]

Schorling suggests that a set of balances is a most useful device for illustrating the meaning of an equation and the application of fundamental axioms for solving equations.[9] He thinks it may be worth while to spend several periods writing equations and illustrating them either with a balance loaded to correspond to the conditions of a given equation or by sketching such a balance on the blackboard. Certainly, if students are to understand equations and if algebra is not to become a mere mechanical shuffling of meaningless symbols, they must not be permitted to fall into the habit of "solving" equations by mechanically "transposing" or "canceling" terms without insight into the reasons why they do it. Moreover, nonmathematical students, because they are less experienced in thinking with and manipulating symbols, are especially prone to develop such habits and even to develop a deceptive facility in juggling symbols unless especial care is taken to make these operations meaningful.

After a student has gained reasonable mastery of the concept of literal numbers and their use in formulas and equations as means of expressing number relations, the concept of signed or directed numbers as a further extension of the number system of arithmetic should be introduced. Some of the more significant problems which will arise in teaching this concept have been discussed previously. Work with signed and literal numbers in general mathematics courses should include application of the four fundamental operations of arithmetic to exercises involving polynomials and algebraic fractions. Emphasis in this work should be, according to the National Committee on Mathematical Requirements, on ability to use ideas, processes, and principles in the solution of practical problems rather than on facility in manipulation.[10]

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[8] *Ibid.*, p. 297.

[9] Schorling, *op. cit.*, p. 149.

[10] National Committee on Mathematical Requirements, *op. cit.*, pp. 14–15. Quoted in *The Place of Mathematics in Secondary Education*, Fifteenth Year-

The algebraic material presented in general mathematics courses usually includes the simpler special products and factor relationships such as the square of a binomial, the product of the sum and difference of the same two numbers, polynomials in which all terms contain a common factor, and trinomials of the form $ax^2 + bx + c$. If work in algebra is to impart increased competence to deal with numerical relationships, this material must be presented in a way which will emphasize the fact that the literal factors studied represent numbers—that these formulas are readily applicable to the numbers of arithmetic. Thus students should be led to see that the square of 63, for example, may be obtained by writing the number as $60 + 3$ and applying the formula for the square of a binomial, or that such products as 57×63 may be readily obtained by writing the numbers as $(60 - 3)(60 + 3)$ and using the formula for $(x - y)(x + y)$. Exercises such as these have a double function. They serve to make clear the meaning of processes and they create interest in mathematics in that through them a student discovers practical application for what he has learned. Too often in the past this application to actual numbers was not made and many students never realized that the letters they were "juggling" represented numbers and that the processes they were so laboriously learning did have significance in everyday computations.

GRAPHS IN GENERAL MATHEMATICS

Probably there is no one mathematical device or technique which is used more often outside of mathematics classrooms than graphical representation of quantitative relationships. No other mathematical device lends itself so well to representation of mathematical data or relationships in a form that is quickly intelligible to the nonmathematician. By the same token, there is no mathematical process or technique which can be more easily misinterpreted or, when used by untrained or unprincipled persons, can so readily mislead or con-

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book of the National Council of Teachers of Mathematics, New York, Bureau of Publications, Teachers College, Columbia University, 1940, p. 87.

fuse. It is evident, therefore, that work with graphs should be an important part of a course in general mathematics.

A student comes to ninth-grade mathematics with a little knowledge of graphical representation. In intuitive geometry he will possibly make bar graphs and circle graphs. But, unless he comes to realize that graphs are but another type of symbolism for expressing number relationships, this work will contribute little to his competence to cope with number relationships. Construction of graphs, even though it gives practice in computation, in measuring, and in working with ratios, and places needed emphasis on neat, accurate work, is of less practical value to a student than practice in reading and interpreting graphs already made. As a consumer of mathematics, a student may have some occasion to construct graphs in order to present to others the relationships existing among certain items. But he will have occasion daily to read and interpret graphs prepared by others. Current newspapers, magazines, financial reports, government bulletins, and books are prolific sources of graphs for students to read and interpret. Particular attention should be paid to graphs that tend to mislead. For example, relative sizes of armies are often compared by relative heights of pictured soldiers. The eye judges their relative size by bulk rather than height. A fairer way, and one which would be less likely to be misinterpreted, would be to present various armies by lines of figures of equal size, the number of figures in each line being proportional to the size of the army it represents. It is far more important that a student be able to read graphs and to spot misleading ones than it is that he be able to construct beautiful ones.

A second important use of graphs is that of providing a means for obtaining approximate solutions to equations relatively easily and quickly. In preparing graphs for this purpose, a student plots pairs of values which satisfy a given functional relation and then draws a smooth curve through the points thus plotted. Intermediate values of either variable may then be read directly from the graph with reasonable accuracy. This particular technique has many practical applications and is frequently used in place of more accurate but slower and more laborious algebraic solutions in business and industry. It has been too much neglected in school work.

CONSUMER MATHEMATICS

Problems in consumer mathematics having to do with topics such as insurance, installment buying, profit and loss, simple and compound interest, taxation, and investments, since they deal with perplexities of vital concern to each of us, are of particular importance in a terminal course in secondary mathematics. Unfortunately, these topics deal with problems of adult concern and, too often in the past, emphasis in work with them has been placed on the computations involved, which are, in most cases, relatively simple. For example, even such a bitter critic of nonfunctional mathematics subject matter as Wrinkle (see pp. 105 ff.) would include a unit on life insurance in the secondary mathematics program. The theory of modern life insurance practice is one of the great triumphs of mathematical theory. But the actual calculation of premium rates and of benefits payable is the work of an actuary, and actuarial science is one of the most specialized and complex branches of higher mathematics. There are few calculations of any practical significance in connection with life insurance that can be performed by ninth-grade students.

Then what are the significant concepts which should receive emphasis in a unit on life insurance at the ninth-grade level? The ideas about life insurance which are of most importance to an individual —the ideas which will serve to make this aspect of his world more intelligible to him—are those relating to the way in which risks are shared, methods of payment of premiums and benefits, types of policies, considerations which should govern choice of policy, the differences between annuities and various types of insurance policies, and the fundamental concepts which underlie a mortality table —in short, the informational aspects of the topic. These concepts can be made intelligible to students of general mathematics.

Much information that is of value in teaching a unit on insurance may be obtained by merely writing letters to insurance companies and asking for it. Gathering and analyzing information about insurance and developing an adequate vocabulary of terms used has many possibilities for fruitful correlation of work in general mathematics and English. Similarly, study of the way in which insurance companies invest their funds, the effect of such great concentrations of capital on the economic life of the nation, and the social implications of life insurance are topics which correlate directly with work in the social studies.

Another common problem met in life is that of installment buying. Now, obviously, the arithmetical problems involved in subtracting the down payment from the total cost of a purchase or in multiplying the periodic payment by the number of payments are too trivial in themselves to warrant inclusion of a unit on installment buying in ninth-grade general mathematics. Moreover, problems of installment buying are adult concerns and it is often difficult to give them any semblance of reality for ninth-grade students. In actual practice, the amounts of the payments in any given transaction are determined from a chart or table and usually neither buyer nor seller knows how these particular figures were obtained.

Examination of textbooks containing material on installment buying reveals that most of them center the work of the unit on problems having to do with the rate of interest paid on installment purchases. Now, is knowledge of the rate of interest paid in a particular instance of major interest to students and teachers, or is insight into relative costs of various payment schedules as compared to cash purchases and into considerations which may lead one to conclude that immediate enjoyment of an article warrants paying the greater cost of a deferred-payment plan, the desired outcome of a ninth-grade study of installment buying? It appears that insight into the many nonmathematical factors which enter into a decision to purchase is at least of equal importance from an educative point of view with the purely arithmetical skills and understandings involved. Evidently, as in the work with insurance, information, rather than practice in computation, is the most valuable outcome to be derived from study of this topic. Yet many textbooks present essentially only a formula which purports to give the rate of interest paid and a group of exercises which can be "solved" by mechanical use of this formula. More than that, some textbooks contain statements about the formula which may be misleading. The formula most often given for finding the rate of interest paid in buying goods on the installment plan is

$$r = \frac{24f}{P(n+1)}$$

where f represents the financing charge, P the amount actually borrowed—i.e., the unpaid balance—n the number of monthly payments to be made, and r the "actual" rate of interest paid. The statement accompanying this formula in one textbook reads: "You can use it (the formula) to find the exact interest rate for *any* installment plan." (Italics mine.) What assumptions are implied in this statement? Whence came the formula in question? Do the conditions which gave rise to it hold in every installment-purchase contract?

The formula

$$r = \frac{24f}{P(n+1)}$$

may be derived as follows:

> Let P be the difference between the cash price of the article purchased and the down payment. It is, therefore, the unpaid balance.
>
> Let f be the cost of financing. It is the difference between the sum of the payments to be made and the unpaid balance P. It may be considered to be the total interest paid.
>
> Let t be the time in years, n the number of *monthly* payments, and r the effective rate at *simple* interest.
>
> Now, if n is the number of monthly payments, then each monthly payment will represent an amount P/n paid off on the principal together

with an interest payment. Payment of monthly installments means, in effect, that the amount P/n was borrowed for 1 month, a like sum P/n was borrowed for 2 months, and so on. The total time t for which the amount P/n was borrowed may be calculated as follows:

$$t = 1/12 + 2/12 + 3/12 + \ldots n/12.$$

This, on inspection, is seen to be an arithmetic progression. Its sum is

$$n/2(1/12 + n/12).$$

Hence,

$$t = n/2(n + 1)/12.$$

Now, if we consider r to be a rate at simple interest, we may write that

$$f = (P/n)(rt).$$

Whence

$$r = \frac{f}{\frac{P}{n}t}.$$

But, as we have seen,

$$t = \frac{n(n+1)}{2(12)}.$$

Whence

$$r = \frac{f}{\frac{P}{\cancel{n}}\frac{\cancel{n}(n+1)}{24}},$$

or

$$r = \frac{24f}{P(n+1)},$$

which is the given formula.

Now, this formula was obtained by assuming that payments would be made monthly and that interest was to be calculated on a simple-interest basis rather than considering that compound interest was being paid. It is quite common for the payment interval on installment-buying contracts to be one week or two weeks, as well as one month, depending on pay-roll practices in a given community. Sometimes the interval will be longer than one month—say, every three months or quarterly. The formula given applies only in case the payment interval is one month and if we assume that simple interest is being paid. In short, the statement quoted above regarding the usefulness of this formula for finding the exact interest rate for *any* installment plan is not quite accurate.

People should realize that they frequently pay dearly for the privilege of buying goods on the installment plan. But is there any merit in countering the ofttimes misleading advertising of companies engaged in installment selling or the small-loan business by equally false or misleading statements? The study of mathematics should impart greater respect for the truth, not be a means of spreading false information. It is true that in the textbook quoted all the exercises given to illustrate the use of this formula were based on monthly payments. But the implication that it applies to *all* installment-buying contracts is not thereby eliminated.

However, the important point in teaching a unit on installment buying is not that excessive rates of interest are sometimes charged. An individual may conclude that immediate possession of an article warrants paying a high rate of interest. He is being victimized only if he thinks he is paying a low rate when it is actually high. It is the function of mathematics education in this instance to provide the insights and information essential to making an intelligent decision, not to render judgment on the wisdom of the practice studied.

This particular topic affords an excellent example of a situation in which the informational content of the subject matter of mathematics has great practical value. The actual calculations are of value only as means of securing real understanding of the concepts involved. From an educational point of view, its value depends, as with all subject matter, on the use which is made of it. If the information and techniques presented are made the means of stimulating students to think critically about the whole problem of credit buying, study of this topic may lead to increased competence to cope with the world and with this problem in particular. But if these materials are used merely for the purpose of presenting propaganda about the high interest rates charged on installment purchases, then real competence to think critically about this life problem or any other will not be an outcome of its study.

ARITHMETIC IN GENERAL MATHEMATICS

There is no lack of evidence that many students enter secondary school without having achieved acceptable standards of accuracy and facility in the fundamental arithmetical operations with integers, common fractions, and decimals. Provision for so-called "remedial work" in arithmetic is given considerable emphasis in most textbooks in general mathematics. However, the routine drill provided is not, in many cases, very effective as a means of raising the level of performance in arithmetic of students enrolled in classes in general mathematics. The reason is not hard to find. To conclude that repetitive drill—with a little reteaching—can do in a few scattered periods during the ninth year what the elementary schools failed to accomplish in eight years is to assume (1) that the student did not have an opportunity to master these arithmetic fundamentals earlier, (2) that the ninth-grade teacher is unusually efficient and effective, or (3) that the elementary teachers were totally incompetent. While any one of these assumptions may be found to apply in individual cases, there is little reason to believe that any or all of them are widely applicable. The student who comes into a ninth-grade class in general mathematics deficient in the fundamentals of arithmetic has probably had ample opportunity to learn the facts and acquire the skills involved. Certainly it would be foolish to assume that ninth-grade teachers as a group are sufficiently better equipped for their task and enough more efficient than elementary teachers in general to be able to accomplish in a relatively short time what the elementary teachers were unable to achieve in eight years. There is, of course, always the possibility that a given student has had the misfortune to do his elementary work in a school which has carried the trend toward deëmphasis of arithmetic to the extreme and so has not had adequate opportunity to learn the facts and acquire the skills of arithmetic. There have been many poorly prepared and incompetent teachers at all levels of our schools, and the students taught by these poor teachers are likely to be academically handicapped as a consequence. But in general, if a ninth-grade student has not mastered the fundamentals of arithmetic by the time he enters the general mathematics class, for some reason he has not wanted to learn them and he will not learn them in the ninth grade unless he can be brought to the point of wanting to learn them.

The stubborn fact remains that many of these ninth-grade students do lack an adequate grasp of arithmetic. There is need at the ninth-grade level for review—or reteaching—of topics such as common fractions, decimals, and percent. The remedial work usually offered fails of its purpose because it is based on a fallacious concept of the nature of arithmetic and an untenable theory of learning. Most remedial drill work in textbooks and workbooks reflects the drill theory of education. It is based on the assumption that the subject matter of arithmetic consists of many specific and essentially unrelated bits of information and mechanical techniques of computation. Learning is assumed to be a matter of habit formation; hence mastery of arithmetic is expected to be achieved by a process of "stamping in" the required responses by repetitive drill. It is assumed that if the student does not have the requisite facility in any specific skill or technique or has not memorized a particular fact additional repetition is the remedy for his failure or neglect.

Serious objections to this drill theory of learning arithmetic are at once evident. In the first place, the task of learning all the separate items which have to be mastered is comparable only to the task of memorizing the symbols of ancient Chinese picture writing.

Brownell cites studies which have been made which underline the utter futility of attempting to master arithmetic in this way.[11] Secondly, as Brownell indicates, repetitive drill is not effective as a means of obtaining the desired mastery of arithmetic.[12] The drill theory ignores the relational-thinking aspect of arithmetic. Arithmetic is not a mass of unrelated facts and skills. As Brownell says, "Arithmetic is best viewed as a system of quantitative thinking," and "instruction through drill does not prepare children for quantitative thinking."[13]

Finally, learning is a process of gaining new insights into relationships, not a matter of stamping in specific responses. If learning were merely a matter of stamping in a specific response, then once a given response had been learned, the individual would always react to the given stimulus by making that specific response. Yet, as Bayles points out:

> If one would predict the behavior of a given individual at a given time, one must know (1) what he wants to achieve, (2) what environment he will face, and (3) what insights or abilities he possesses. Given *only* a certain environmental stimulus, one is not in a position to say what a given individual will do when that stimulus impinges on his nervous system. . . . A host of responses is possible, and at a particular time the *most appropriate* response is likely to occur, not the response which previously has been most often practiced. Goal, environment, insight—all three must be known and properly interrelated before prediction is possible. [Italics in original.][14]

Action in any situation will depend, therefore, on the individual's insight into relationships existing between his goal and the environment. Learning is a matter of sensing new relationships—of developing new insights. If this be true, teaching should be a matter of stimulating a pupil to discover new relationships, not one of supervising the practice of old reactions.

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[11] William A. Brownell, "Psychological Considerations in the Learning and the Teaching of Arithmetic," *The Teaching of Arithmetic*, Tenth Yearbook of the National Council of Teachers of Mathematics, New York, Bureau of Publications, Teachers College, Columbia University, 1935, pp. 6–7.

[12] *Ibid.*, pp. 8–10.

[13] *Ibid.*, p. 10.

[14] Ernest E. Bayles, "Drill, or Thrill, in Education?" *Elementary School Journal*, 40:24–36 (September, 1939).

But, as Bayles says, "Another way of saying that we have insight into something is to say that we have caught its *meaning*. And meaning means essentially pointing. To catch the meaning of something is to realize what it points to, or signifies. Thus to say that learning is a process of developing insight is to say that learning is a matter of catching new meanings."[15] Evidently, then, remedial drill in arithmetic must be drill which emphasizes the meanings of the facts and techniques of arithmetic in such manner that the student is able to grasp the relationships which exist among these facts and techniques. And the student will learn to use these facts and techniques only when he sees that these relationships do afford him a simpler means of attaining his own purposes—when he sees that they have meaning in terms of the situation which confronts him. Brownell says, "The 'meaning' theory conceives of arithmetic as a closely knit system of understandable ideas, principles, and processes. According to this theory, the test of learning is not mere mechanical facility in 'figuring.' The true test is an intelligent grasp upon number relations and the ability to deal with arithmetical situations with proper comprehension of their mathematical as well as their practical significance."[16]

How can the facts and techniques of arithmetic be presented to ninth-grade students in a way that will make them want to discover what meaning these facts and techniques have for them? The answer is to be found in the history of the origin of all mathematical theory. Mathematics was developed as a means of solving specific problems which confronted individuals and for which they were concerned with finding a solution. So, if the material of arithmetic can be presented to the ninth-grade student through the medium of problems which are real to him and for which he is concerned about finding solutions, he will want to master the requisite facts and techniques and ninth-grade teachers may actually accomplish in a relatively short time what the elementary school failed to do.

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[15] Ernest E. Bayles, *The Theory and Practice of Teaching*, New York, Harper & Brothers, 1950, p. 67.

[16] Brownell, *op. cit.*, p. 19.

The work in algebra and in intuitive geometry included in courses in general mathematics will afford much practice in arithmetical computation and, properly presented, will be fruitful of opportunity to develop new insights into number and space relationships. Many problems involving common fractions arise in connection with practical problems in the shop. The less academically inclined students in general mathematics classes are likely to have had sufficient experience with such problems to appreciate that they do offer a perplexity which it would be to their advantage to resolve. The work with percents likewise can be focused on and motivated by the practical applications of percent and the theory presented as a means of resolving difficulties with which the students have become familiar through their out-of-school experiences. Drill on the fundamentals may very well come from simple exercises in keeping accounts, reconciling bank statements with checkbook stubs, or computing the standings of athletic teams. If students once grasp the insight that the techniques of arithmetic are, in fact, useful techniques for solving their problems, there will be no difficulty in getting them to attempt to master them.

SUMMARY

In short, a ninth-grade course in general mathematics should represent an attempt to utilize the subject matter of mathematics to teach worth-while lessons to students whose academic abilities or interests in mathematics are too limited to enable them to profit from a study of the more traditional courses. The aim is not mastery of specific subject matter so much as it is the development of desirable habits of thinking about problem situations arising from their environment and, in particular, greater competence in dealing with problems arising out of the quantitative aspects of their daily living.

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Chapter 9

Other Courses in Secondary Mathematics

THE traditional sequence of mathematics courses in the secondary school has usually been algebra in the ninth year, demonstrative geometry in the tenth year, a half-year each of advanced algebra and solid geometry in the eleventh year, and, if a four-year sequence in mathematics was offered, a half-year of trigonometry and another half-year of algebra in the twelfth year. For students who are not interested in college-preparatory mathematics, courses in commercial arithmetic or in shop mathematics, or an advanced course variously termed senior mathematics, consumer mathematics, or something similar, might be offered. In recent years the tendency has been in the direction of making a course in general mathematics the typical ninth-grade course. Students interested in taking more work in mathematics then take algebra in the tenth grade and plane geometry in the eleventh. Occasionally a third semester of algebra and a half-year of trigonometry or of solid geometry will be offered in the twelfth year.

It is becoming increasingly common for students to seek entrance into colleges and universities with only one year of work in secondary mathematics; more often than not, that is a course in general mathematics rather than the more conventional algebra. Many times a course labeled algebra will resemble a general mathematics course more closely in spirit and in content than it does the traditional algebra. Even though engineering schools, for example, list solid geometry as a prerequisite for entrance, increasing numbers of high-school graduates are seeking entrance into college without credit in that course. Colleges, therefore, are finding it necessary to offer courses in what is essentially secondary-school mathematics to care for the needs of those students whose secondary-school preparation in the subject is inadequate in terms of their college objectives. A student who has a strong interest in the fields for which more extensive preparation in mathematics is essential must, therefore, delay work in his chosen field until his deficiencies are made up. Others, unwilling to mark time while they are catching up this neglected preparatory work, simply choose new objectives which do not have such specific mathematical prerequisites. For a college-preparatory student, therefore, a secondary-school mathematics program going beyond the traditional one-year-each of algebra and geometry is desirable if his later educational and vocational choices are not to be unduly restricted.

However, four out of five of the students enrolled in our high schools probably will not go to college. For that reason, the secondary-school program in mathematics must be planned with the needs of this non-college group in mind. Colleges can, and will, provide the necessary preliminary courses for those students whose secondary-school preparation in mathematics proves inadequate in light of their college objectives. After all, these courses were originally a part of the college curriculum.

But if mathematics has any real place in secondary education other than as a means of meeting college entrance requirements, it is especially important that its values be made available to non-college-preparatory students since they will not have an opportunity to make up their deficiencies later in college. There is no evidence to indicate that one year of work in general mathematics or even a year of algebra followed by a year of demonstrative geometry will provide an individual with all the skills and understandings needed to cope with this modern world or that any one or two years of work in mathematics will exhaust its potentialities as a means of developing ability and willingness to do critical, reflection-level thinking about the problems of life for most secondary-school students. It is certainly true that the applications of mathematics to new aspects of life are becoming more extensive every day. More and more occupations and professions formerly not considered mathematical are coming to demand personnel with more or less extensive training in mathematics.

All secondary-school pupils should know the extent of the mathematical training required for entrance into various professions, business careers, or industrial positions. They should also have an adequate knowledge of the cultural and social significance of mathematics. It is certain that serious handicaps to later educational or vocational success often result from omission or postponement of mathematical training, that the time a student should spend in study of mathematics should be determined by the extent to which he can profit from that study, and that capable students should be given an opportunity to continue their study of mathematics throughout the entire secondary-school period. The conclusion follows, therefore, that "It is undesirable to regard any high school year prior to the last as a 'terminal' year for mathematical study, and to crowd into it numerous unrelated topics simply to make sure that the pupils will have encountered them. Pupils need to meet important topics repeatedly, on successively higher levels. Maturity also is requisite for the proper understanding of portions of the subject. It is therefore necessary to have a mathematical curriculum that runs through several years according to a systematic plan."[1]

There is good evidence, moreover, to support the contention of the Joint Commission that "In the case of retarded pupils, it is impossible at any stage to crowd into a single year all the necessary remedial work and also the usual program for that year."[2] Hence, even for nonmathematical students, there may be need for further work beyond the typical one-year course in general mathematics. Advanced courses, in the case of nonmathematical students, would probably be weighted heavily in the direction of commercial arithmetic and shop mathematics.

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[1] *The Place of Mathematics in Secondary Education,* Fifteenth Yearbook of the National Council of Teachers of Mathematics, New York, Bureau of Publications, Teachers College, Columbia University, 1940, pp. 73–74.

Numerous variations in the traditional advanced-mathematics sequence at the secondary level are possible. The Joint Commission in its report outlines two plans for providing a unified course in mathematics from the seventh grade through the twelfth.[3] Whether the traditional sequence is retained or some plan similar to those suggested by the Joint Commission is adopted will depend on local needs and resources. However, because of the sequential nature of mathematics, it is likely that any satisfactory plan will, for the most part, make use of the traditional subject matter. So, in what follows, the discussion will center around the time-honored divisions of subject matter as a matter of convenience.

ADVANCED ALGEBRA

In the traditional sequence, a course in advanced algebra follows ninth-grade algebra and tenth-grade demonstrative geometry. Since students taking this course will have had considerable mathematical experience and will be relatively homogeneous as to ability and interests—weak or disinterested students will probably not elect this course—the subject matter considered can be more difficult and the treatment more rigorous than is possible in ninth-grade algebra.

However, even though the students electing advanced algebra will be, in general, those who were superior students in ninth-grade algebra, usually it will be found necessary to begin the work of the advanced course with a thorough review of the basic concepts and operations of the elementary course. Care will need to be taken to keep this review from being so extensive that it becomes boresome

[2] *Ibid.*, p. 76.

[3] *Ibid.*, pp. 72–119.

and so destroys the interest of the class at the start. Frequently, inexperienced teachers make the mistake of spending so much time reviewing that little time is left to devote to new ideas and the advanced algebra course becomes merely a "refresher" course in ninth-grade algebra. This situation, of course, should be avoided.

A more important question has to do with the need for review. Why should superior students who should have attained a better than average understanding of elementary algebra need any review of basic concepts before attacking advanced work in the subject? Why will not use of these fundamental concepts in the new work provide the necessary review?

In the first place, in order to make satisfactory progress in more advanced work it is essential that a class be able to perform fundamental operations with signed and literal numbers with accuracy and facility and that they be reasonably proficient in factoring and in work with algebraic fractions. Even though these operations are clearly understood, some review is necessary to bring back lost skill in using them. In addition, however, there will often be misconceptions and difficulties which need clarification. For example, as Butler and Wren indicate, much of the difficulty which students have with algebra stems from lack of understanding of some minor point.[4] Thus students may actually not realize that $\frac{1}{3}x$ and $\frac{x}{3}$ are equivalent. In factoring, often the difficulty with an expression like $(27x^6 + 64y^9)$ is that they cannot see it as $(3x^2)^3 + (4y^3)^3$. As Butler and Wren suggest, students need to be taught specifically to recognize that type formulas for factoring may be applied in such instances.[5] Work with special products and factoring in advanced algebra should differ from that of ninth-grade algebra largely in the greater emphasis given to this type of exercise. Particular emphasis should be given to factoring exercises in which the elements are themselves binomials, such as $6(x + y)^2 - 13(x + y) + 6$, or higher powers of an unknown, as $3x^6 + 8x^3y^4 - 3y^8$, or involve literal exponents, as $3x^{2a} + 5x^a + 2$. Often such exercises serve to bring to light misconceptions relative to operations with algebraic symbols. Seemingly, it is difficult for students to realize that procedures they have learned to use in working with monomial factors and terms and with integral numerical coefficients and exponents apply equally well when the factors and terms are binomials or polynomials and when the coefficients and exponents are literal numbers or fractions. Realization of this fact will help greatly in developing a realization of the generality of algebraic operations.

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[4] Charles H. Butler and F. Lynwood Wren, *The Teaching of Secondary Mathematics*, New York, McGraw-Hill Book Co., 1941, pp. 341–342.

[5] *Ibid.*

It will be well to introduce some additional type forms also. A student probably will not have had experience at the ninth-grade level with such forms as the sum or the difference of two cubes; with expressions of the form $x^4 + ax^2y^2 + y^4$, which may be reduced to the difference of two squares; or with more complex trinomials of the form $ax^2 + bx + c$. However, a word of caution needs to be introduced here. There is a strong temptation to spend too much time in drill on such topics because students seem to gain facility in factoring and in work with exponents and radicals very slowly. Fortunately, the more complicated forms are infrequently used and the later work with fractions and with quadratic equations will give ample additional practice in factoring and with the exponential notation.

Study of operations with exponents and radicals should be extended to include fractional, negative, and literal exponents, and techniques for manipulating radicals. Logarithms and the slide rule may be presented as applications of the theory of exponents.

Study of linear and quadratic functions is an important part of an advanced algebra course. There is usually need for some review work in solving fractional equations, but the most extensive consideration should be given to solution of quadratic equations. The solution by factoring is, of course, fundamental. Here the emphasis must be on the reasoning which permits us to set each factor equal to zero and solve. Students must understand that this procedure is possible only because of the fact that when the product of two factors is zero, one or the other of those two factors must be zero. Quadratics which are not readily factorable may be solved by completing the square or by use of the quadratic formula. The method of completing the square is of value in algebra chiefly because it is the means by which the quadratic formula is derived.

Before presenting the process of completing the square, preliminary work should develop need for a method which is applicable when the left member is not readily factorable—possibly through the use of a seemingly simple problem of a type which the students have been solving by factoring but which turns out to be nonfactorable. Perhaps the best method of presenting the process itself is through use of a geometric diagram. If the example used in presenting the process for the first time is a quadratic which is factorable, it is a simple matter to verify the correctness of the result obtained by the already familiar method of factoring. This also serves to emphasize the fact that the new method is completely general, that it applies to *all* quadratics. Suppose we are to solve the equation $x^2 + 4x - 12 = 0$ by the method of completing the square. We may proceed as follows:

STEP 1. Transform the equation to the form

$$x^2 + 4x = 12.$$

STEP 2. Suggest that the situation may be represented pictorially by a rectangle of known area but of unknown dimensions. We then have

$$x^2 + 4x = 12. \qquad \text{(see Fig. 9a)}$$

STEP 3. Evidently this figure may be divided into a square that is x units on a side and a rectangle that is 4 units by x units.

$$x^2 + 4x = 12 \qquad \text{(see Fig. 9b)}$$

STEP 4. The smaller rectangle may be divided into half lengthwise and the outer half moved to a new position without altering the total area.

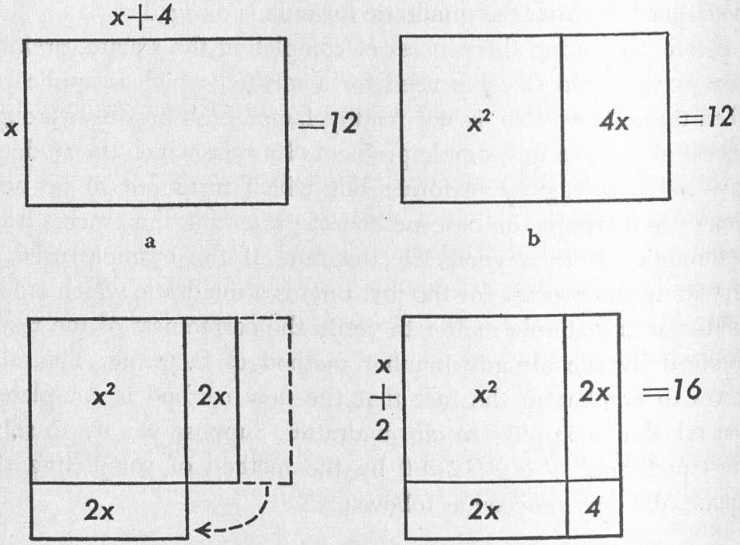
$$x^2 + 4x = 12 \qquad \text{(see Fig. 9c)}$$

STEP 5. If we "complete the square" by filling in the corner, we shall have added a square of known size and shall then have a perfect square of known area.

$$x^2 + 4x + 4 = 12 + 4$$

(see Fig. 9d)

$$(x + 2)^2 = 16$$



a b

c d

FIG. 9

STEP 6. We may find the value of the length of one side of this square by taking the square root of the area. Algebraically, we then have

$$x + 2 = \pm 4.$$

STEP 7. Solving the *two* linear equations which result will give the values of x.

$$x + 2 = +4 \qquad x + 2 = -4$$
$$x = +2 \qquad x = -6$$

These values may be verified by factoring the original equation. This process should be repeated using different equations, some of which are factorable and some of which are not. Then the steps for application to any trinomial should be formulated. Students should solve a sufficient number of examples using the method of completing the square to become reasonably proficient with it and to recognize that it will, in fact, give the solution of any quadratic. The quadratic formula should then be developed as the general solution of all quadratics of the form

$$ax^2 + bx + c = 0$$

by solving this general quadratic for x by the method of completing the square.

Unfortunately, too many students never seem to realize that the quadratic formula can be applied to equations with literal coefficients other than the traditional a, b, and c of the general quadratic as usually written or to equations with fractional or literal exponents. Many algebra students become quite proficient in using the quadratic formula to solve equations of the form

$$6x^2 + 9x - 15 = 0$$

even at the ninth-grade level. Yet science teachers have almost ceased to expect that these same students will be able to solve the formula

$$\frac{gt^2}{2} - vt = s$$

for t correctly. Why is it apparently so difficult for a student who is proficient in using the quadratic formula to solve equations in x or y with numerical coefficients to recognize that it may be used equally well to solve

$$e^{2x} + 5e^x - 7 = 0$$

for e^x?

Obviously, such students have not mastered the idea that the quadratic formula is a *general* solution for *all* quadratics *and for all equations in quadratic form.* Apparently, for many students, insight into the meaning of the formula is limited to the concept that it may be used to solve all second-degree equations with numerical coefficients.

Suppose we consider the equations:

$$ax^2 + bx + c = 0$$
$$6x^2 + 9x - 15 = 0$$
$$\frac{gt^2}{2} - vt = s$$
$$e^{2x} + 5e^x - 7 = 0$$
$$3(x - y)^2 + 5(x - y) + 9 = 0$$

Is there any relationship of coefficients and variables common to all of these equations? Inspection will reveal that in every one the first term contains a factor which is the square of a factor of the middle term and which is not present in the third term. If students can be led to realize that this, and not the values of the coefficients, is the distinguishing characteristic of a quadratic, it will not be difficult for them to grasp the idea that the x of the general equation $ax^2 + bx + c = 0$ may represent a t, e^x, $x - y$, or any other algebraic expression equally well. Once this insight is established, application of the formula to equations in the quadratic form will readily be made.

Obviously, it is too much to expect students to generalize the formula to the extent that they can apply it to unfamiliar forms without some guidance. Therefore, to insure their gaining the desired insight into the nature and potential utility of the quadratic formula, extensive practice in applying it to equations which do not appear at first glance to be quadratics and to equations in the quadratic form should be given.

Similarly, the use of the discriminant in determining the nature of the roots of a quadratic will need to be pointed out and practice in using it provided. Work in finding the sum and product of the roots of a quadratic is of value in developing a clear idea of the relationships expressed by the quadratic formula quite apart from and beyond the uses made of these concepts in more advanced mathematics. For example, is the quadratic formula merely a device for finding the value of x in a particular type of equation or does it have other practical applications? There is, of course, a reciprocal relationship between the factors of a quadratic function that can be factored into rational, integral factors and the roots of the equation formed by setting that function equal to zero. That is, if $x = 3$ and $x = 5$ are roots of the quadratic equation $x^2 - 8x + 15 = 0$, then $x - 3$ and $x - 5$ are factors of the quadratic function $x^2 - 8x + 15$. In light of this reciprocal relationship between the roots of a quadratic equation and the factors of the function, how can the quadratic formula be used to find the factors of a quadratic function which is not readily factorable by inspection? A skillful teacher will not find it difficult to lead students to discover that, if the roots of $3x^2 - 7x + 3 = 0$ are $x = \frac{7 + \sqrt{13}}{6}$ and $x = \frac{7 - \sqrt{13}}{6}$,

$$3x^2 - 7x + 3 = \left(x - \frac{7 + \sqrt{13}}{6}\right)\left(x - \frac{7 - \sqrt{13}}{6}\right).$$

A student who has gained this insight into the theory of quadratics is much more likely to recognize that a given function is quadratic in a binomial or exponential expression than one who thinks of the quadratic formula as merely a relationship of the numerical coefficients of equations of the form $ax^2 + bx + c = 0$. The student who has grasped this wider insight into the quadratic relationship will have gained not only added competence in mathematics but a heightened appreciation of the value of generalizations and of the way in which concepts acquired from previous experiences may be transformed and modified to cope with new situations.

Not only should a student gain facility in solving equations but he should also become proficient in solving verbal problems which require setting up an equation for solution. Why are students who can solve a given linear or quadratic equation readily so often completely at a loss when given a problem leading to an equation and asked to set up the equation and solve it? What insight must a student possess to be able to use algebraic theory in solving practical problems? Does the student who can solve a given linear or quadratic equation readily but who is unable to solve a problem leading to such an equation really understand the theory of linear or quadratic equations? If he cannot write an equation leading to the solution of a given problem he has not fully grasped the idea that algebraic expressions stand for numbers and that algebraic equations are simply statements of number relationships. To him, algebraic operations can be nothing more than meaningless juggling of abstract symbols. But to one who has grasped the idea that algebraic symbols are generalized number concepts, the expression of the conditions of a problem in an equation is simply a matter of restating the given relationships using the symbolic language of algebra. Proficiency in solving verbal problems is both a necessary and a sufficient indication of mastery of the theory of algebra.

Solution of systems of linear equations in two and three unknowns and the simpler cases of systems involving second-degree equations may well be included in a secondary-school advanced algebra course. Some time should also be devoted to work with the simpler types of equations involving radicals. Graphical solution of systems of linear or of quadratic equations in two unknowns affords an excellent means of introducing the basic concepts of analytic geometry and for emphasizing the utility and wide applicability of rectangular coördinate systems. Graphs of equations, either linear or of higher degree, afford a method of finding approximate roots with reasonable accuracy and ease. Because of its practical applications, this aspect of the graphical solution needs considerable emphasis. Graphical methods not only provide secondary-school students with a feasible method of approximating real roots of equations of higher degree but, particularly in connection with solutions of systems of equations, help them to visualize the significance of the algebraic solutions.

Work with quadratic equations will not progress far before need for the basic concepts of complex numbers will be felt. For example, the equation $x^2 + 3x + 12 = 0$ does not seem to differ in any important respect from other equations which may be solved readily by the quadratic formula. But in this instance, application of the formula indicates that $x = \frac{-3 + \sqrt{-39}}{3}$ or $x = \frac{-3 - \sqrt{-39}}{3}$. A brief review of the theory of radicals will serve to indicate to a student that here he is facing a dilemma. By definition, $\sqrt{n}$ indicates one of the two equal factors of n. But if factors are equal they must have the same sign and the product of two factors of like sign is positive. Therefore, a negative number cannot be the product of two equal factors. Yet, in the problem under consideration, the value found for x involves finding the two equal factors of -39. How can this dilemma be resolved? As he is casting about for a way out of this impasse, the student may be reminded that to eliminate restrictions on subtraction it was necessary to invent a new type of number—the negative number. He will readily agree that subsequent work in algebra has demonstrated the utility, convenience, and "reality" of negative numbers. With this example in mind, students will come to accept the idea of *inventing* a new number i, defined as being equal to the unknown $\sqrt{-1}$, as a way out of the difficulty. However, they need to understand clearly that the term *imaginary* as applied to these numbers is a misnomer, that these numbers *exist* as surely as do positive and negative numbers but were originally termed imaginary because they were not part of the then known number realm, that in their applications to electrical theory, for example, they are as *real* as any of the numbers of the *real* number system. It may be helpful in making this clear to point out that negative numbers were once termed "fictitious numbers" but that as they became better known and were better understood their "reality" was no longer questioned. Students will readily see the utility of using i to simplify such problems as $\sqrt{-4}$ by writing it as $\pm 2i$ and, when they grasp the idea that complex numbers are simply new symbols devised to cope with problems met in particular phases of science and engineering, they will have no further doubts about their "reality." Incidentally, development of the values of the powers of i as the recurring series $i, -1, -i, +1, i, -1, -i, +1, \ldots$ affords an excellent test of the extent to which students can think in terms of algebraic symbols. Similarly, graphical work with complex numbers will help to demonstrate the value and use of the vector concept in a totally new field.

Problems in compound interest and with simple annuities also lead directly to work in arithmetic and geometric progressions and to the binomial theorem. These have so many practical applications in business and in everyday concerns that they should be studied rather thoroughly. If time permits, work with permutations and combinations and with the mathematical concept of probability will have immediate application to the field of insurance and to games of chance. Student interest in these topics is always high.

The aim in teaching advanced algebra, whether the individual student intends to go on to college and take additional work in mathematics or to complete his formal education in the secondary school, is, as in ninth-grade algebra, to provide a further acquaintance with a most useful instrument for developing and expressing ideas about the world as we find it. Proficiency in manipulating symbols is not its primary purpose. Obviously, a student must achieve a reasonable mastery of the techniques of algebra to be able to make any use of it either to understand the ideas others are trying to convey or to present his own ideas to others. But mere facility in the operations of algebra is not the goal. The goal is to bring students to realize that algebra is a generalization of arithmetic, that the processes and procedures of algebra are means of working with ideas.

Teaching methods used will, therefore, stress understanding rather than mechanical facility. But, in addition, increased competence to use the techniques of algebra in further work in mathematics or in practical situations must be given adequate consideration. The student who cannot use a concept does not understand it.

TRIGONOMETRY

Formal trigonometry in the sense that conscious use is made of the trigonometric functions by name is commonly studied only in the later years of the secondary school, if then. But the fundamental concepts of trigonometry as indirect measurement using the properties of similar triangles are used successfully even in the later elementary-school years. Since trigonometry, whether taught as a formal course by that name or as a unit in some other mathematical course in which the fundamental concepts of the formal course are utilized, has considerable practical utility, is not too difficult for fairly young pupils, uses concepts drawn from both algebra and geometry, and affords many opportunities for generalization and for developing skill in analysis and in reflective thinking, its study can contribute materially to the development of an independent thinker competent to cope with the modern world. The study of trigonometry should be, therefore, an integral part of the secondary-school curriculum in mathematics.

Possibly the first introduction to the concepts of trigonometry may come at the seventh- or eighth-grade level in solving problems having to do with measurement of inaccessible distances by means of similar triangles. At this level, such problems as determining the height of a tree by measuring the length of its shadow and the length of the shadow cast by some object of known height at the same instant provide experience in using the basic concepts of trigonometry even though the functions used may not be mentioned by name. It is not necessary, or even desirable, to introduce the terminology of the formal course at this level. The work may well be confined to exercises involving measurements in the field, problems solved by scale drawings, and a brief treatment of the numerical solution of such problems by proportion.

At ninth-grade level a similar unit may be used to introduce the idea that the ratios of pairs of sides in a right triangle are functions of the angles. A class may be led to discover this fact experimentally by measuring the sides of various right triangles having acute angles of fixed size and then computing the ratios. After students discover by actual measurement that the ratios of particular pairs of sides are constant for a given angle, crude tables may be worked out from these measurements and used in working simple problems. At this point the convenience of a uniform terminology for indicating which ratio is being considered will become evident, and the terms *sine, cosine,* and *tangent* may be introduced. Sometimes only one of these—often the tangent—is presented.

Trigonometry as taught at junior-high-school level will be intuitive in character—i.e., based on concepts established by measurement and experiment rather than by logical proof—and will stress application of basic concepts to practical problems. In the tenth grade, in connection with work on similar triangles in geometry, the theory of the trigonometric functions can be studied. The concept of the reciprocal relationships of the functions may be introduced at this time and also the relations existing between functions of complementary angles. The use of simple tables may be introduced in the tenth grade and all the functions used in solving problems. On completing the tenth grade, a student should have a fairly extensive knowledge of trigonometric theory as applied to right triangles and be able to use these concepts in simple problems.

The formal course in trigonometry offered at the eleventh- or twelfth-year level of the secondary school will differ but little from a college course in trigonometry. Frequently the same textbooks are used, and students taking the course in high school may or may not be required to repeat it in college. There is a pronounced tendency, however, for the high-school course to give greater emphasis to solution of oblique triangles than is given in college classes and less emphasis to work with trigonometric identities, conditional equations, and the theory of the general angle. Young, for example, lists the following eight topics as making up the subject matter of trigonometry:

1. The definition of the ratios, the use of tables of their natural values, and applications to problems involving the right triangle.
2. The use and theory of logarithms.
3. The solution of general triangles with and without logarithms.
4. The fundamental relations between the ratios and simple identities.
5. The general angle and extension of the definitions of the ratios and identities to apply to the general angle.
6. Formulas for expressing the functions of the general angle in terms of functions of a reference angle plus or minus a multiple of 90°—i.e., $\sin(270° - x) = -\cos x$.
7. Trigonometric equations.
8. The formulas having to do with the functions of the sum or difference of two angles.[6]

He suggests that the first three or four of these topics are well within the reach of a secondary-school pupil but that the last four may well be left for college-level work.[7]

The Joint Commission recommends that a secondary-school course in trigonometry include:

1. Definition of the six functions and the appropriate reduction formulas.
2. The basic identities involving a single angle.
3. The addition formulas for the sine, cosine, and tangent and the double-angle and half-angle formulas derived from the addition formulas.
4. The law of sines, the law of cosines, and the law of tangents, with applications of each.
5. Simple identities and trigonometric equations.
6. Field work.[8]

The Commission suggests that the addition formulas, the double-angle and half-angle formulas, and the law of tangents may, if necessary, be deferred until the twelfth year.[9] The actual content of the course taught will depend, of course, upon local conditions, requirements of state courses of study, and similar considerations.

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[6] J. W. A. Young, *The Teaching of Mathematics in the Elementary and the Secondary School,* New York, Longmans, Green and Co., 1925, pp. 288–289.

[7] *Ibid.*

[8] *The Place of Mathematics in Secondary Education,* Fifteenth Yearbook of the National Council of Teachers of Mathematics, pp. 95–96.

[9] *Ibid.*

The first step in beginning any course in formal trigonometry is, of course, to learn the names of the six functions and to get clearly in mind what they mean. The most obvious approach is by measurement of corresponding sides of similar right triangles. At eleventh- and twelfth-grade level, however, it is quite likely that a student will already be familiar with the concept of sine, tangent, and perhaps cosine through his earlier work in algebra and geometry. If not, he will be familiar with the properties of similar triangles and the six ratios may be developed in terms of the ratios of line segments as applied to any right triangle. It will usually be found better to insist on using the phrasing "side opposite the angle *A* divided by the hypotenuse" in defining the sine function of *A*, for example, than to permit a student to fall into the habit of saying that the sine of *A* is "*a* over *c*." (See Fig. 10.) Apparently students have difficulty in picking out the proper line segments in triangles which are not lettered in the standard way unless they learn the definitions in terms of "side opposite the angle," "side adjacent to the angle," and "hypotenuse." It is also well to make sure that students understand that the hypotenuse is the side opposite the right angle. If a teacher will insist on use of these expressions until the habit is formed, and provide practice in application to triangles with vertices lettered other than the conventional *A*, *B*, and *C*, students will learn to think of the ratios in these terms and will have less difficulty in making applications later on.

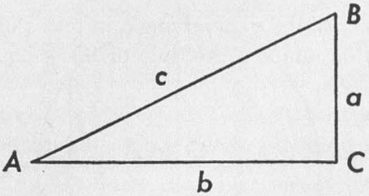

FIG. 10

There is a trend, in the more recent books on trigonometry at the college level, to begin with the concept of the general angle and to define the functions at the beginning in terms of "ordinate," "abscissa," and "radius vector." (See Fig. 11.) Certainly one terminology is no more difficult than the other, and students who have sufficient mathematical maturity to be studying formal trigonometry should be familiar with the terms *ordinate, abscissa,* and *radius vector.* If the definitions of the ratios are first learned in terms of a general angle, there will be no need to modify or to relearn them later on.

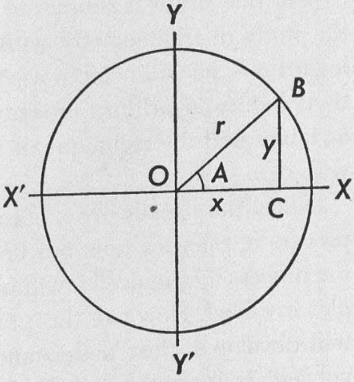

FIG. 11

Whichever method of introducing the definitions of the functions is adopted, it will usually be found convenient to introduce the functions of complementary angles in connection with an explanation of the exact meaning of the terms *cosine, cotangent,* and *cosecant.*

After the names and definitions of the functions are fairly well established, their values for angles of 30°, 45°, and 60° should be learned. Seemingly, any student ought to be able to pick out these relationships from the appropriate triangle without difficulty, yet college students sometimes do have trouble in working them out. However, these angles and their functions are used so frequently that, like the multiplication tables, the values of their functions should be memorized if necessary and students should be able without hesitation to sketch triangles showing the angles and their corresponding sides in proper proportion.

Learning to use tables of natural functions and of logarithms is a fundamental part of the study of trigonometry. Considerable practice is usually required to be able to use tables and to interpolate readily and accurately. If students have already learned to use tables of logarithms, they have little difficulty in learning the additional skills involved in using tables of trigonometric functions. But for those who are learning to use tables for the first time, interpolation in either a table of mantissas or a table of trigonometric functions is beset with difficulties. A table of mantissas, since it does not exhibit the confusing phenomenon of sometimes increasing with increasing values of the function, as does the sine, and at other times decreasing with increasing values of the variable, as in the case of the cosine, is simpler to learn to use. But if a student begins the study of trigonometry without having learned to use tables of logarithms, he will need to use tables of natural trigonometric functions before logarithms are introduced. In either case, proper use of tables and the technique of interpolation will need to be taught with care.

Difficulties usually arise, not from any inherent complexity in the process of interpolation, but because students attempt to carry out the process mechanically, without clear understanding of the principles involved. How can this process be presented so that a student will develop a clear understanding of the ideas and principles involved? Is it anything more than a mechanical procedure? What are the concepts which may cause confusion?

Probably the primary cause of confusion is to be found in the fact that values given in a table of trigonometric functions or a table of logarithms are not exact. Students have probably not realized, when they first come to work with tables, that such instances as the sine of 30° which is exactly .5 or the tangent of 45° which is exactly unity are rare exceptions to the general rule that tabular values are approximate. Therefore, they see no way of finding the value of, for example, a function of an angle which is not given exact in the table. Once a student is led to discover that the tabular values are approximate and that the variation in them is always uniform for small changes in the angle, he will readily recognize that interpolation is essentially a matter of solving a problem in proportion. Moreover, this insight will also serve to dispel the confusion arising from the second conflict—that some trigonometric functions increase in value and others decrease. The fundamental concept of interpolation as a problem in proportion holds for a decreasing function as well as for one that increases. The modification required is at once evident to a student who sees clearly the relationship between the tabulated values and the value he seeks. And, once he understands that in any situation interpolation is simply a problem in proportion, he should be able to perform the necessary computations mentally and, with a little practice, to use tables of proportional parts with facility and confidence.

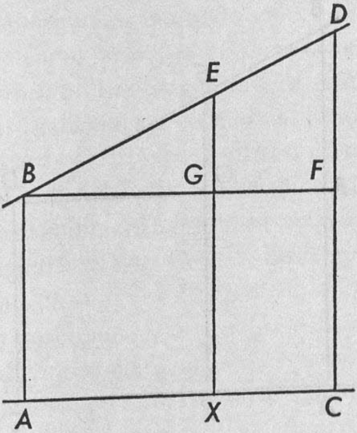

FIG. 12

The process by which a student may be led to develop the required insights into the interpolation process can best be presented by means of numerical examples. Suppose we assume that the problem is to find the natural sine of 26° 31.4′ using five-place tables. A student readily finds that sin 26° 31′ = .44646 and that sin 26° 32′ = .44672. Obviously, the sine of 26° 31.4′ must lie somewhere between these values. It is not difficult for him to see that, if AB (see Fig. 12) represents the sine of 26° 31′ and CD represents the sine of 26° 32′, the difference between CD and AB may be represented by the segment FD, and that GE, the difference between AB and XE, may represent the difference between the sine of 26° 31′ and the unknown sine of 26° 31.4′. It is also quite readily evident that $\triangle BGE$ is similar to $\triangle BFD$, hence that $GE/BG = FD/BF$. Numerically, in the example given, this means that $DF = .44672 - .44646$ or .00026. $BF = 1'$ and $BG = .4'$. We have, therefore, that $GE = .4(.00026)/1$ or .000104. Since $XE = AB + GE$, it is evident that XE is equal to .44646 + .00010, dropping the last digit from the value found for GE. Hence we may write that the sine of 26° 31.4′ is .44656. It is well to emphasize, in explaining why the terminal digit was dropped from the value found for GE, that in this process we have assumed straight-line variation of the sine function between 26° 31′ and 26° 32′, which is contrary to fact but, as an approximation, is sufficiently accurate for practical purposes. If this fact is not made clear, students have a tendency to carry additional digits in interpolation, thinking that they will thereby achieve greater accuracy.

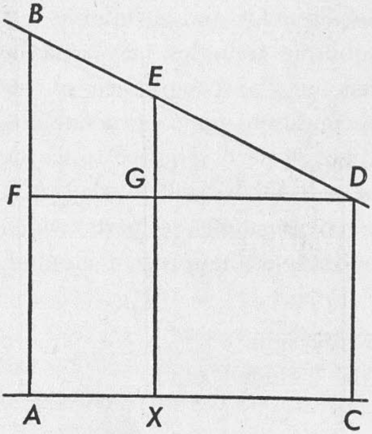

Fig. 13

The modification of the above process needed to interpolate for seconds instead of tenths of a minute is readily evident. If the function sought is a decreasing function, such as the cosine or cotangent, the procedure required is not so readily evident. If the appropriate figure is lettered as before (see Fig. 13), CD, which represents the tabular value of the greater angle, will be less than AB, representing the tabular value of the smaller angle. Suppose, for example, that we wish to find the cos 36° 42.6′. We find from the table that cos 36° 42′ = .80178 and that cos 36° 43′ = .80160. Evidently GE, the difference between AB and CD, is equal to .00018. As before, $\triangle DGE$ and $\triangle DFB$ are similar and $GE/DG = FB/DF$. But the side DG of the smaller triangle is proportional to $1' - .6'$ whereas AX, which represents the difference between 36° 42′ and 36° 42.6′, has the same ratio to AC or to FD that .6′ has to 1′. So we must calculate GE in terms of the distance from X to C. We have, then, $GE = .4(.00018)/1$ or .000072. If, as before, we round off this value to five places and add it to the value of CD, we obtain a value of .80167 for XE, from which we conclude that cos 36° 42.6′ is equal to .80167. It would have been equally feasible, of course, to have calculated XE as equal to $AB - .6(AB - CD)$. Butler and Wren suggest that it may be helpful to use simple numerical problems to illustrate the principle of interpolation before attempting to interpolate in tables.[10] Once a student has gained insight into the geometrical relationships involved, he will have little difficulty in performing the required interpolations quickly and accurately.

Problems involving solution of oblique triangles may well be introduced by considering theorems on the congruence of triangles. If it be pointed out that the conditions for congruency are, in effect, statements of the combinations of parts required to fix the size and shape of a triangle, then it will not be difficult to lead a class to discover that the possible combinations of parts which exactly fix the size and shape of a triangle are three in number. They are:

Case 1. Given two angles and any side.
Case 2. Given two sides and the included angle.
Case 3. Given the three sides.

There is, of course, in addition to these, the ambiguous case in which we are given two sides and an angle opposite one of them and which may or may not have a unique solution.

The law of sines may readily be developed as a means of solving triangles in which the conditions given are those of case 1. However, students tend to be confused by the form in which this law is usually given:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Hence, a teacher should take pains to see that students recognize that this form is merely a convenient way of indicating the relationships:

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[10] Butler and Wren, *op. cit.*, pp. 437 ff.

$$\frac{a}{\sin A} = \frac{b}{\sin B}; \qquad \frac{a}{\sin A} = \frac{c}{\sin C}; \qquad \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Moreover, students should be led to see that each of these may be written in several different forms as:

$$\text{(a) } \frac{a}{\sin A} = \frac{b}{\sin B}; \qquad \text{(b) } a = \frac{b \sin A}{\sin B}; \qquad \text{(c) } \frac{a \sin B}{\sin A} = b;$$

$$\text{(d) } \sin B = \frac{b \sin A}{a}; \qquad \text{and (e) } \sin A = \frac{a \sin B}{b}.$$

Obviously, each of these can be obtained from the original form by simple algebraic transformations, but students at the secondary level are not always sufficiently experienced in operations with trigonometric functions to realize that they can be manipulated algebraically.

The forms (d) and (e) above, in which the quantity sought is the sine of an unknown angle, and hence, indirectly, the angle itself, lead to the ambiguous case. Two insights are essential to an understanding of this case. In the first place, the student must understand why it is that knowing the sine of an angle does not enable him at once to fix uniquely the value of that angle. In the second place, he needs to see for himself that one, two, or no solutions may be possible from the data given. If he is led to discover these latter possibilities by means of scale drawings and if the reason for the possible ambiguity in the value of the angle corresponding to a particular value of the sine function is carefully developed, the ambiguous case will not offer particular difficulty. It does, however, have value as a means of bringing about realization of the meaning of the values of functions of angles greater than 90° all out of proportion to its practical usefulness as a means of solving triangles.

The solution of triangles involving case 2 relationships may be accomplished, of course, by use of either the law of cosines or the law of tangents. The law of cosines, since it is not adapted to calculations with logarithms, is of value chiefly as a basis for developing the half-angle formulas. Therefore, a student should learn to depend primarily on the law of tangents for solving problems involving two sides and the included angle. The law of cosines or the half-angle formulas may be used for solving problems under case 3 conditions. Again, the law of cosines, because it is not adapted to the use of logarithms, is best used merely to develop the half-angle formulas, and solution of applied problems in which three sides of a triangle are given is best accomplished by use of the half-angle formulas.

In solving problems involving applications of trigonometry, a student should learn to follow a definite form in arranging his work. Most textbooks suggest a form, and a teacher should insist that it, or some other satisfactory form, be used. If a student gets the habit of organizing his work in orderly fashion, of making a sketch showing the conditions of the problem, and of carefully labeling each logarithm or other number used in the solution, he will save time in preparing his lessons and at the same time gain a heightened appreciation of the practical value of habits of neatness, order, and accuracy. Unfortunately, students often fail to gain an insight into the practical value of orderly, systematic, and precise habits of working and thinking from their study of mathematics because they are permitted to be slovenly in doing their daily preparation and written work.

Radians and mils, as units of angular measurement, like the work with identities and trigonometric equations, offer difficulties to a high-school student because they represent ideas for which he has no background of experience or need. Like the units of the metric system, the radian and the mil represent alternate units for a system of measures which is familiar to the pupil and which seems to serve his needs very well. The only real reason why the concept of radian should be introduced into the trigonometry course is that it is needed in more advanced work in mathematics and in science. The mil, of course, is used chiefly by the armed forces although it has many characteristics to commend it for peacetime use as the basic unit of angular measurement. However, if the geometric bases for the radian and the mil are carefully presented, pupils may gain from them an insight into the true nature of any unit of measurement and of angular measure in particular which is difficult to acquire in any other way. And the radian is used so often in advanced work in mathematics and science that some familiarity with it is desirable even if a student does not expect to pursue his study of mathematics to subjects such as calculus where its use is sometimes imperative. Hence, the concept of radian measure should be studied carefully and pupils given opportunity to become reasonably proficient in thinking of angles in terms of radian units. They should be able to translate from degrees to radians and from radians to degrees with facility and certainty. They should be able to recognize the special angles such as 30°, 45°, 60°, 90°, and 180° when expressed in radian measure with the same facility that they recognize ¼ and 25 percent as equivalent.

Trigonometric identities and trigonometric equations offer the same type of difficulty to students at the secondary level that mils and radians do. They have had little reason to use these relationships in problems they wish to solve. The principal applications of these ideas are to be found in more advanced work in mathematics and science. However, work with identities and with trigonometric equations affords valuable experience in using algebraic techniques in a new way. If the consequences of each proved identity and each solved equation are carefully brought out, work with trigonometric identities and equations will serve to broaden and clarify a student's insight into the nature and interrelationships of the trigonometric functions—and of the generality of algebraic techniques.

The chief obstacles encountered in work with identities and with trigonometric equations stem from confusion as to the distinction between an equation and an identity and from the consequent failure to recognize the techniques which are appropriate in each case. After students have learned to recognize that if a relationship is true for all values of an angle it is an identity, but that if it holds only for specific values of the angle it is an equation of condition, then they will have little trouble in recognizing that they can prove an identity by reducing the members independently to equivalent expressions while they can solve an equation of condition by use of such algebraic techniques as clearing of fractions, adding or subtracting the same quantities from both members, and so on. A second difficulty with both types of problem lies in the fact that it is somewhat hard for a student to learn to make appropriate substitutions. He is a bit hesitant to substitute some other function for the given item such as replacing $\sec^2 \alpha$ by $(\tan^2 \alpha + 1)$ even though he knows that the two are exactly equivalent. He has had little occasion or opportunity to use this technique in previous work in mathematics and it does not seem quite valid to him. Moreover, he needs experience to determine readily which substitutions will be advantageous and which will not. However, the familiarity that comes with practice will overcome these difficulties.

Applications of trigonometry to science, surveying, navigation, and astronomy are so numerous and so full of interest in themselves that motivation is not a problem in teaching trigonometry. In fact, because there is this wealth of interesting and practical applications and because trigonometry offers so many opportunities to use, review, and extend concepts of arithmetic, algebra, and geometry at a level of difficulty well suited to the mathematical maturity of secondary-school students, there is real danger that teachers will become so engrossed in teaching these techniques and applications that they will lose sight of the wider educational values that should be gained from study of trigonometry.

Any school activity—and a mathematics class in particular—should be a problem-solving experience that leaves a student better prepared to deal with the perplexities of everyday living. Merely mastering the techniques of solving triangles does not, except in a few cases, add appreciably to an individual's competence to cope with his world. But if, at the time a student is mastering the techniques of solving triangles, he comes to realize that the methods used exemplify a thinking process with more general application, then, as a consequence of his study of trigonometry, he will become more competent to discover and resolve inadequacies and disharmonies in his outlook on life. The subject matter of trigonometry offers many opportunities for reflective studies of concepts that appear to be inconsistent—as, for example, when trigonometric functions defined in terms of sides and angles of right triangles are used to solve oblique triangles or applied to angles greater than a right angle. To the extent that a teacher can capitalize on pupil interest in the problem situations of trigonometry to stimulate growth in competence to solve the problems of everyday living, study of trigonometry will serve to achieve the objectives of secondary education in a democratic society. But always the educational values that may be derived from its study represent potentialities to be realized, not goods inherent in the subject matter.

SOLID GEOMETRY

Solid geometry is coming to be somewhat of a no man's land in mathematics in that the number of high schools which offer courses in it is decreasing from year to year while college departments of mathematics are hesitant to offer it for college credit. Many colleges, particularly schools of engineering, require it for entrance but seldom provide any opportunity for students who did not take it in high school to make up their deficiency in the subject. Yet, as Breslich suggests, we live in a three-dimensional world, so that study of the geometry of three dimensions would seem to have practical significance.[11] Certainly, all of us have rather frequent opportunity to exercise spatial imagination in determining whether packages will fit into certain spaces, in arranging and rearranging furniture, or in placing stock on shelves. The history of mathematics seems to indicate that the first geometric relationships which man recognized were three-dimensional. Two-dimensional geometry is, in reality, an abstraction from the geometry of three dimensions.

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[11] Ernst R. Breslich, *Problems in the Teaching of Secondary Mathematics*, Chicago, University of Chicago Press, 1930, p. 311.

One reason for the decrease in number of schools offering solid geometry as a separate course is the tendency to combine plane and solid geometry into a single course. This is not a particularly new idea. According to Young, as early as 1900 a paper was presented before the International Congress of Mathematicians at Paris advocating that, whenever convenient, the theorems of two-dimensional space should at once be extended to the comparable relationships in three-dimensional space.[12] There has always been considerable work in mensuration of three-dimensional objects taught informally at the elementary level. It is certainly possible to present many of the significant relationships informally.

The Joint Commission lists the following "significant and sometimes forgotten" merits of the study of solid geometry:

> 1. The traditional course in solid geometry offers an opportunity to enlarge the pupil's grasp of postulational thinking by a more thorough treatment of converses, inverses, and contrapositives of theorems, of necessary and sufficient conditions, and of indirect proofs.
> 2. Solid geometry offers opportunity for extensive applications of the algebraic theory of radicals, rationalization of denominators, radical equations, and the manipulation of complicated algebraic expressions.
> 3. It offers opportunity for computational work with logarithms.
> 4. Drawing figures and constructing models in solid geometry offers worthwhile experience in analyzing and representing spatial relationships.
> 5. Work on loci in space and the difficulties involved in mensuration of three-dimensional figures afford opportunities for further experience with the power of the mathematical method to overcome obstacles.[13]

But these merits urged by the Joint Commission are not peculiar to three-dimensional geometry. Does study of solid geometry make significant contributions to a student's insights into the relationships of three-dimensional space? Certainly an individual needs wide acquaintance with the subject matter of solid geometry. The concepts of spherical geometry are essential to an understanding of geography, of navigation, of astronomy, and of modern physics.

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[12] J. W. A. Young, *op. cit.*, pp. 285–286.

[13] Adapted from the report of the Joint Commission in the Fifteenth Yearbook of the National Council of Teachers of Mathematics, *The Place of Mathematics in Secondary Education*, pp. 115–116.

Spatial factors are ever determining factors in any physical relationship. However, the factual information in the course in solid geometry can be, and is, taught informally. In view of the crowded curriculum in modern secondary education it is doubtful whether study of solid geometry contributes a sufficient number of unique opportunities for developing a student's ability to reason reflectively to warrant its being taught for that reason alone. Hence, separate courses in solid geometry will probably continue to decrease in number and the real values to be gained from its study will be sought through informal study of the subject matter and in the tenth-grade course in demonstrative plane geometry.

If, in a particular school situation, it is neither desirable nor practical to offer a separate course in solid geometry, care should be taken in the course in demonstrative plane geometry to call attention to the analogies which exist between perpendiculars and parallels to lines in a plane and to planes in space, between congruence and similarity relationships in plane and solid figures and in plane and spherical polygons, between plane and dihedral angles, and between circles and spheres.

OTHER COURSES IN SECONDARY MATHEMATICS

If the contention of the Joint Commission that "It is undesirable to regard any high school year prior to the last as a 'terminal' year for mathematical study" be accepted, then it is necessary to plan a four-year curriculum in secondary mathematics.[14] For an individual who likes mathematics, whether or not he expects to continue it in college, the usual secondary-school sequence of algebra, plane geometry, advanced algebra, trigonometry, and solid geometry still leaves one semester of the senior year in which he might take an additional course in mathematics if he so desired. And if the solid geometry course is combined with the plane geometry course in the tenth year, a whole year may be available for additional work in mathematics.

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[14] *Ibid.*, p. 73.

The most obvious solution is to offer an additional course in algebra which is often, for want of a better name, called college algebra. The Joint Commission suggests that, if this alternative is chosen, "the work of this last semester should be essentially that of the regular course in college algebra and the standards of performance be such that the pupil will be able to omit the college course in algebra."[15] A second possibility suggested by the Joint Commission is a course designed to acquaint pupils with the elements of analytic geometry, the calculus, mathematics of finance, and part of elementary statistics.[16] Such a course might well be a full-year course and would be even more valuable to a student who did not expect to go to college than to a college-preparatory student. It would give the former an acquaintance with the techniques and concepts of advanced mathematics which would be most informative and valuable. It would provide a college-preparatory student orientation in mathematics which would help to motivate much subsequent work in college mathematics. At least one textbook designed to provide such a course is available.[17]

However, the more common pattern in secondary school is not to offer four years of work in mathematics except in the larger schools and even there for only a small number of students. The more typical case is that of a student who takes general mathematics in the ninth year and who does not take algebra or geometry. Or he may take one of them and not the other after having completed the course in general mathematics. The Joint Commission recommends a course designed to acquaint such pupils with certain problems of modern society from a quantitative point of view.[18] The following quotation from the preface of a recent textbook indicates the point of view and purposes of such a course: "This book provides a new course in mathematics. It does not replace anything in the traditional mathematics offering of a high school or junior college; rather it is an addition and certainly an enrichment. It comes to grips with some of the problems of daily living that confront most of our families as consumers—something that the standard sequential courses of the past half century have not done and cannot do."[19]

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[15] *Ibid.*, p. 118. [16] *Ibid.*

[17] Virgil S. Mallory and Howard F. Fehr, *Senior Mathematics for High Schools,* Chicago, Benj. H. Sanborn and Co., 1940.

[18] *The Place of Mathematics in Secondary Education,* pp. 116–117.

The Joint Commission suggests that such a course should include the following topics:

1. Measurement and computations with emphasis on approximate numbers, significant figures, degree of accuracy, logarithms, and short ways of multiplying and dividing.
2. The necessary review of the fundamental operations with common fractions, decimals, and percents, together with practice in organizing and presenting problems that may involve only arithmetic but that are more difficult than are those treated in the ninth grade.
3. The simpler concepts of statistical methods in the construction of graphs, frequency tables, and scatter diagrams, and the calculation of measures of central tendency, dispersion, and correlation.
4. The construction and use of index numbers in connection with such topics as real wages, commodity prices, cost of living, and business cycles.
5. The problem of making and operating a household budget at various income levels.
6. The problems of installment buying including rate of interest paid, factors entering into the charges made, and the advantages and disadvantages of buying on credit.
7. The mathematics of investments in stocks, bonds, and annuities, banking, home owning, periodic accumulations, and the amortization of debts.
8. Problems of insurance, both property and personal.
9. The mathematical aspects of taxation, crop control, social security, foreign exchange, and price fixing.[20]

In teaching a course of this kind, no attempt would be made to respect traditional subject-matter boundaries. The course would not meet college-entrance requirements or serve as a prerequisite for additional work in mathematics or the sciences. It would represent an attempt to help students grow in competence to solve the problems of everyday living rather than an attempt to teach a specific body of subject matter. Emphasis would be on information rather than manipulation although, of course, the quantitative aspects of the topics studied would be stressed. For example, in the study of statistics, calculation of the arithmetic mean and the median would be taught. But emphasis would be on the use that can legitimately be made of such measures rather than on the process of computing them. It is more important that a student be able to interpret a statistical graph than that he be able to prepare one. So pupils in such courses would make graphs only in order to understand how they may be used to present—or mispresent—data. To quote again from Schorling, Clark, and Lankford:

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[19] Raleigh Schorling, John R. Clark, and Francis G. Lankford, Jr., *Mathematics for the Consumer*, Yonkers, New York, World Book Co., 1950, p. iii.

[20] Adapted from *The Place of Mathematics in Secondary Education*, p. 117.

> There is no implication that algebra and geometry are not important. They have always been and still are a necessary or desirable part of the training of many persons, especially those who wish to enter the field of science or mathematics. . . .
>
> It is a tragic mistake to assume that the ordinary citizen will need no more mathematics than did his great-grandparents. Today the mathematical needs of our citizens go beyond mere mechanical skill in the fundamental operations. There are a few basic mathematical ideas and principles without which man can neither read nor think intelligently about the ordinary affairs of daily living; without which he cannot expect to attain functional competence in mathematics.[21]

It is the function of a course in socioeconomic mathematics such as has been described to provide this functional competence for the majority of the students enrolled in the secondary school.

Courses in business arithmetic and in shop mathematics are sometimes taught at the secondary level. Their content is, as would be expected, that material which will be of greatest direct value to students wishing specific training in the mathematics of business or shop. Such courses offer no unusual teaching problems other than that there is probably greater emphasis on facility in computation than is usual in more general courses. However, it is a mistake to sacrifice understanding of fundamental principles for mere speed in computation, as is sometimes done.

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[21] Schorling, Clark, and Lankford, *op. cit.*, p. v.

After all, whatever the purpose in studying mathematics, it is primarily a way of thinking, and any teaching procedure which neglects to emphasize this aspect of the subject will not achieve the full potentialities of the material to which it is applied.

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PART III

Methods, Evaluation, and Preparation for Teaching Secondary-School Mathematics

Chapter 10

Classroom Management and General Teaching Method

THE classroom is the place where the theory and the practice of teaching meet. In discussing theories about the proper way to teach mathematics in secondary schools our attention is, and should be, focused on such factors as precise statement of the goals to be achieved, formulation of criteria for choice of subject matter and of teaching method, and means of evaluating the progress made. But when we attempt to put these theories into practice in the classroom, we must also take into account such factors as lighting, ventilation, proper seating, and such routine administrative details as taking the roll and keeping records. Thus, teaching methods that are effective, feasible, and desirable with a class of ten may prove entirely unsuited to a class of forty—or a most desirable procedure may require resources in the way of material equipment that are simply not available. Time spent in taking the roll or in passing out and collecting instructional supplies and papers cannot be used for teaching mathematics. A well-chosen teaching procedure may fail because of a poorly ventilated room or because of disorderly conditions growing out of disciplinary problems. Effective teaching requires effective planning and efficient management of all factors of the total teaching situation as well as wise choice of subject matter and of teaching method.

PROBLEMS OF TEACHING METHOD

If the study of mathematics is to be a means of developing ability to do critical, reflection-level thinking about all types of problems, then it would seem that it should be taught by techniques which emphasize such thinking. How can such teaching be done? Suppose each new topic were to be introduced as a problem situation, a situation presenting a dilemma the way out of which is not apparent? Suppose, instead of presenting a solution ready-made, the teacher should lead his class through the steps by which a solution may be obtained? Would not the exercises worked by pupils then serve to help them fix in mind the method of thinking used in arriving at a solution? Is it not reasonable to assume that, if the pupil is to do the thinking rather than the teacher and if mathematical concepts and techniques are to be something more than "machines" in which pupils put numbers and crank out "answers," it must be the pupils who are active during the teaching and learning process? If so, is not the problem of method one of devising ways and means to stimulate desirable mental activity on the part of students? Would we not define as desirable mental activities which result in growth in problem-solving ability—activities which lead to increased willingness and ability on the part of pupils to do critical, reflection-level thinking?

Bayles terms the kind of teaching which promotes reflection-level thinking *reflective teaching.*[1] He says:

> Each teaching unit should represent a reflective study of challenging problems. . . . This means that subject matters which are settled, so far as a given class is concerned, are not proper instructional material. Before teaching can really get under way, a given matter must become unsettled for the class. No reflective study can get under way until a student has reached the "I don't know" stage, nor will it proceed satisfactorily unless the student is concerned with knowing. Therefore an essential feature of a teacher's stock in trade is the ability to maneuver a

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[1] Ernest E. Bayles, *The Theory and Practice of Teaching,* New York, Harper & Brothers, 1950, p. 137.

class into wanting to obtain an answer to a question which it cannot at first answer.[2]

To many persons, at first glance, the problem-raising tactic of teaching advocated by Bayles seems ill adapted to instruction in mathematics since mathematics, to a greater degree than any other field of learning, is characterized by the relative certainty of the conclusions reached and, except at the most advanced levels, by the lack of controversy over the techniques used. But can we not emphasize that mathematical conclusions are certain only to the extent that there is agreement as to the premises on which they are based? Have not many of the techniques and concepts of mathematics been developed as means of finding ways to reconcile seeming inconsistencies and contradictions? Is not a given procedure often only one of several possible alternatives? Mathematics has benefited from the efforts both of those who sought means of solving problems of immediate practical concern and of those who were intrigued by apparent inconsistencies and inadequacies in abstract theories. However, in final analysis, the techniques and concepts of mathematics were developed as a consequence of the efforts of individuals who had encountered "I don't know" situations which they were determined to resolve. Unfortunately, the mathematical theory in textbooks represents only the final results of their efforts to solve their problems. All the debris of creative thinking, the false starts and other evidence of the way in which the final insight into the desired solution was gained have been polished away. The so-called "problems" in a typical mathematics textbook are not problems in the sense of being ideas in conflict; they are simply practice exercises. As Bayles says, "The pupils have already been told how to go about handling matters of this kind. They do not have to consider various possible ways to achieve solution and thereby decide which way to use. They have already been told what to do and they need only to go ahead and do it. They are not faced with a forked-road or no-road situation. They have been told, 'This is the way to go; now go.' Therefore, precision in word usage would require that such practice materials should be called exercises."[3]

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[2] Ernest E. Bayles, "The Relativity Principle as Applied to Teaching," *University of Kansas Bulletin of Education*, 4:11 (February, 1940).

This statement does not mean that pupils should not be required to solve exercises of this type. After a given procedure or technique has been developed reflectively, they should solve a sufficient number of these practice exercises to give facility in its use and to insure that they fully understand its applications and significance. By facility we mean that a student has mastered a concept or technique to the extent that he can use it skillfully and confidently. Often students have difficulty with new concepts, not from any lack of ability to grasp the insights involved, but because they lack sufficient skill and precision in using techniques needed in their development.

However, a teacher should first present a new concept or technique in a true problem situation. For example, as indicated in an earlier chapter, the concept of directed numbers may very well be introduced by calling attention to the lack of agreement between actual practice and the arithmetic ban on subtraction of a larger number from a smaller. Actually, in this instance, the problem raised is less one involving a conflict of ideas than it is a matter of bringing about a realization of the convenience—and the possibility—of extending the number system of arithmetic. It does, however, present students with a difficulty to be resolved and, skillfully handled by a teacher, may well serve to initiate a reflective teaching process. Similarly, other concepts may be introduced through the medium of practical problems which require use of new techniques for their solution or which may be solved more readily by use of these new ideas. The essential idea is that a student see a particular concept or theory as a means of obtaining the solution to a problem that is real to him. In geometry, for example, reflective teaching would require that study of a theorem begin with a question about the relationships which exist in the particular situation being considered. The statement of the theorem to be proved would be one of the steps in the total reflective process called forth by this question—not the starting point. Thus, after the congruence relationships of triangles having two sides and the included angle or two angles and the included side respectively equal have been established, either as postulates or by formal proof, a teacher might raise the question as to whether any combination of three parts was sufficient to fix a triangle. A reflective study seeking to establish the truth or falsity of this proposal should result in (1) discovery and formal proof of the theorem that triangles having three sides respectively equal are congruent, (2) discovery of the ambiguous case in which triangles having two sides and an angle opposite one of them respectively equal may or may not be congruent, and (3) establishment of the fact the angles alone do not fix the size of a triangle. But the proof of the theorem would be the final activity of the teaching process, not its inception.

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[3] Ernest E. Bayles, *The Theory and Practice of Teaching*, p. 188.

In short, reflective study requires that instruction start with a question which cannot be immediately answered and which therefore constitutes a problem. Reflective teaching begins with such a problem and leads the student, through study, to a solution.

One of the most effective classroom techniques for reflective teaching of mathematics is that of directed or supervised study. If a long explanation of a new process is given before the student is asked to do anything about it, he will often be unable to grasp the full import of the technique or process because he has not as yet experienced a situation in which there was a need for it. The idea looks easy as the teacher presents it, so a student tends to sit back relaxed, only to discover later that he has not really understood it at all. But if a teacher starts with a problem, or problems, to be solved and suggests that certain material in the text will be useful in finding out how to do the work assigned and then sets his students to work on the assignment, a different situation obtains. As soon as a pupil thinks he can solve the first problem—or exercise—he starts in. If he has really understood the process as explained in the text, he will be able to work exercises illustrating it without difficulty. But, as is often the case, if he has not grasped the full significance of the theory presented, he soon comes to a point where he does not know how to proceed. Now, when he realizes that here is a new situation, one with which he is not prepared to cope, he is ready to seek suggestions about ways to resolve his difficulty. The problem has become real to him and he is ready to carry out a reflective thinking process which will culminate in an understanding of the new concept. So, in using the directed-study procedure, a teacher does not take the center of the stage but moves about the room, aiding each student as he needs help and permitting each pupil to progress at his own speed. If a particular problem is causing difficulty, as soon as several students have discovered a difficulty in it the teacher may call the whole class to listen to a discussion of the particular point involved.

A modification of the directed-study procedure is that in which part of the class is sent to the board, assuming that there is not sufficient blackboard space available for the whole class to work there at one time, and put to work on problems dictated by the teacher, or at developing formulas, making constructions, proving theorems—whatever the work in hand for the day may be. The part of the class not at the board should do the same work on paper at their seats. The advantage of having some pupils work at the board is that a teacher can check the work of several students at a glance and see quickly who is making satisfactory progress and who needs extra help or if additional explanation or practice drill is desirable for the whole group. This technique may be used in developing new processes or formulas such as the quadratic formula. By questioning and suggestion, the teacher leads his pupils to do the thinking which is, of course, the whole purpose of the exercise.

It is always undesirable for students to attempt something too far beyond their powers. They become discouraged. But if the methods suggested above are used, a student gains confidence in his ability to solve problems and to interpret the text. He learns by actual practice the way to attack a problem, in the mathematics classroom and out of it. The assignment that should follow a directed-study lesson will be a follow-up and completion of the work started under the teacher's supervision rather than an attack on new work.

READING IN MATHEMATICS

There is considerable evidence indicating that inability to read mathematical subject matter with facility and understanding is a prolific cause of low achievement and of failure in mathematics at the secondary level and in college. As Georges points out, "Training in reading is a part of the elementary school program but it does not necessarily follow that pupils entering the high school are competent to use reading as a study tool."[4] Terry found that adults classified as decidedly better than average with respect to reading rate read ordinary expository prose at significantly higher rates than they read either arithmetical problems or isolated numerals.[5] He concluded that "Arithmetical problems and isolated numerals are decidedly more difficult as types of reading material than ordinary prose."[6] Leary says that the method and the rate of reading mathematical materials are quite different from the method and the rate of reading fiction and popular nonfiction because mathematics "utilizes material in which the facts are closely packed together and intricately interwoven."[7] She suggests also that mathematical matter is "more meticulous, more careful, more fatiguing, and hence 'harder' for the average student" than nonmathematical material.[8] And, since mathematical matter requires a type of reading technique different from that called for by the subject matter of an English course and since, as Georges suggests, "High school pupils need further training in the use of reading as a study tool and a differentiated type of training in order to meet the demands made on them," must we not agree with his further conclusion that "It is apparent that the department of English alone cannot assume the whole responsibility for training in reading"?[9]

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[4] J. S. Georges, "The Nature of Difficulties Encountered in Reading Mathematics," *School Review*, 37:217–226 (March, 1929).

[5] Paul W. Terry, *How Numerals Are Read*, Supplementary Educational Monographs No. 18, Chicago, University of Chicago, 1922, p. 97.

[6] *Ibid.*

[7] Bernice E. Leary, "Meeting Specific Reading Problems in the Content Fields," *Forty-Seventh Yearbook of the National Society for the Study of Education, Part II, Reading in the High School and College*, Chicago, University of Chicago Press, 1948, p. 150.

[8] *Ibid.*

This conclusion would seem to imply that teachers of mathematics are to assume the responsibility for giving training in the type of reading required of pupils in mathematics classes. If so, what are the specific reading disabilities common to students of mathematics and how should we go about increasing reading efficiency in so far as mathematical subject matter is concerned? Georges found that approximately 37 percent of all reading difficulties in algebra were caused by the vocabulary and symbolism of mathematics and that inability to interpret the author's illustrative material because of inadequate background knowledge of mathematical processes and relationships accounted for another 21 percent.[10] He found that lack of intensity in reading was the cause of approximately 13 percent of all reading difficulties in algebra. Under this heading he included inability to read descriptive textual material well enough to assimilate its meaning readily and inability to grasp the full meaning of a statement. He reported that difficulties due to lack of precision in reading accounted for an additional 11 percent of all reading difficulties in algebra. Under this heading he listed (1) omission of parts of statements, (2) inaccurate reading of familiar phrases, (3) solving for quantities other than those called for in the exercise, (4) failure to look up references mentioned in the statement of the problem, and (5) incomplete statements due to careless reading. Georges found that difficulties arising from failures on the part of the pupils to analyze problems correctly —such as, for example, failure to select the facts necessary for the solution of a problem or inability to associate textual explanations with geometrical figures—were the causes of trouble about 10 percent of the time, while difficulties arising from the manner in which the exercise was stated in the text accounted for the remaining 8 percent of the reading difficulties in algebra studied by him.[11]

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[9] Georges, *op. cit.*, pp. 217–226.

[10] *Ibid.*

From the standpoint of a teacher of secondary mathematics, the various types of reading difficulties reported by Georges may be considered under three heads. First, difficulties caused by the vocabulary and symbolism of mathematics and by the lack of an adequate background of mathematical knowledge to interpret correctly what is read are essentially those growing out of a lack of knowledge of the subject matter of mathematics. Second, difficulties due to a lack of intensity or of precision in reading and to failure to analyze problems correctly are more directly due to faulty reading techniques. And third, difficulties arising from the manner in which the exercise is stated do not have their primary origin in the subject matter of mathematics or in faulty reading techniques; they arise from lack of experience with mathematics on the part of the pupil and from lack of clarity or from use of vocabulary and grammatical structure ill adapted to the maturity of the learner or from carelessness on the part of the teacher or the textbook writer.

The first requisite to effective reading in mathematics, as in any subject-matter field, is mastery of the basic techniques of reading. Leary and Gray are of the opinion that it is reasonable to assume that by the time boys and girls reach high school and college they have acquired the basic habits and skills on which mastery of the "mechanics" of reading depends.[12] They warn, however, that "Not all pupils acquire expertness in the mechanical processes of reading in the elementary grades. Many continue through school misreading information and making false interpretations chiefly because they are too absorbed in identifying new words to get meaning. The only safe attitude, therefore, is to regard basic habits as possible sources of difficulty at any level and to provide training whenever there is evidence that they have been neglected."[13] It is the responsibility of a teacher of mathematics to discover these problem cases and, if possible, to refer them to reading specialists for remedial work. If no such expert assistance is available, he should seek suggestions as to desirable procedures from the literature on general techniques for teaching reading. Often much can be accomplished by enlisting the aid and coöperation of teachers in other areas, English in particular. In any case, the problem is primarily one of teaching reading; the student's difficulties with mathematical subject matter are a consequence of his inadequacy in the basic reading skills rather than the mathematical material's being a cause of his reading difficulties.

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[11] *Ibid.*

[12] Bernice E. Leary and William S. Gray, "Reading Problems in Content Fields," *Reading in General Education*, William S. Gray, ed., Washington, American Council on Education, 1948, p. 116.

The reading problems of greatest concern to a teacher of mathematics are those which have their origin in the special form and content of mathematical subject matter. If we accept the results found by Georges as typical of the frequency of occurrence of the various types of reading difficulties experienced by secondary-school students in mathematics, the most promising way to improve performance in mathematics by means of improved reading techniques must be through development by the student of a more adequate knowledge and understanding of the vocabulary, symbolism, and factual content of mathematics since almost 60 percent of the difficulties studied by Georges had their origin in these aspects of mathematical subject matter.

Vocabulary difficulties in mathematics are of two types. On the one hand are the unfamiliar technical terms such as *exponent, coefficient, angle, plane, coördinate, quadratic, binomial,* and *discriminant* which must become a part of the working vocabulary of a student if mathematics is to have any meaning for him. In the same category are such expressions as *collect like terms, graph an equation, clear of fractions,* and *drop a perpendicular,* and unfamiliar words from science, industry, and the business world found in problem material. Possibly more confusing, on the other hand, are such familiar words as *right, normal, rational,* and *radical* which have specialized meanings in mathematics. Available evidence indicates, according to Leary, that most students need definite instruction in the vocabulary of mathematics and that only a minority will develop the requisite understanding and technical vocabulary through regular classwork in mathematics.[14] Hemphill found that "Most pupils in beginning algebra will improve in ability to solve verbal problems if, under the supervision of their classroom teachers, they effectively study the vocabulary of their text and selected reading exercises that involve mathematical material."[15] He adds that "Pupils of superior mental ability seem more likely to benefit by such study procedures than pupils of average or low mental ability."[16] As a first step in achieving an adequate mastery of the vocabulary of mathematics, students should be made aware of the limitations of their mathematical vocabulary. This step may well be achieved by means of a short, informal test on the meanings of terms selected from the daily assignment. Secondly, students should be encouraged to check new words whose meaning is unknown or doubtful and to follow up with an intensive study of the words marked as unfamiliar. Strang suggests that a student "write each word on a card, with its derivation, definition, and a sentence in which the word is used correctly. These cards may be used for daily drills until the word has become a permanent part of the student's vocabulary. In order to develop versatility of attack on unknown words, the drill described in the foregoing sentence should be supplemented by instruction in methods of sounding new words, and of analyzing and sensing their meanings from context."[17] Formal drill on technical vocabulary may be used, particularly in beginning new work. Gray recommends that the vocabulary training provided "should include emphasis on all aids to work recognition, namely: (a) context clues which are of great value in inferring both meaning and pronunciation; (b) word-form clues, including both general configurations and significant details; (c) structural analysis, which involves a scrutiny of words for clues to their spoken counterparts; (d) phonetic analysis, which makes use of the sounds of letters and groups of letters in recognizing words; and (e) the dictionary which is an essential aid in promoting independence in word recognition."[18] In carrying out these steps, there must be constant emphasis on the special meanings which common words have in mathematics and on the application of these general techniques to mathematical subject matter.

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[13] *Ibid.*, pp. 116–117.

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[14] Bernice E. Leary, *op. cit.*, p. 158.

[15] Reid Hemphill, "Improving Linguistic Ability as a Factor in Solving Problems in Algebra," *Abstracts of Doctoral Dissertations in Education,* Kansas Studies in Education, Vol. II, No. 6, Lawrence, Kansas, University of Kansas Publications, 1941, p. 25.

[16] *Ibid.*

[17] Ruth Strang, "Diagnosis and Remediation," *Reading in General Education,* William S. Gray, ed., Washington, The American Council on Education, 1948, p. 336.

A second source of difficulty in reading mathematical subject matter has its origin in the symbolism of mathematics. Hogben has likened mathematics to a language,[19] and Dantzig uses the same concept in the title of his book.[20] Thorndike declared that "The mere knowledge of the language of algebra has more utility than educators have thought while skill in computing has less,"[21] and Leary points out that "Today, one of the major objectives in mathematics in the secondary school is the development of an understanding of a new mathematical language, which begins with the language of algebra."[22] She continues, "The elements of this language include (1) alphabetical symbols or literal numbers, (2) operational symbols, (3) directed numbers, and (4) methods of showing relationship—formula, equation, table of values, and graph. Each of these elements has its peculiar difficulties for the beginning pupil."[23] Whether or not a teacher of mathematics follows Hogben's example and presents these elements as the parts of speech of the new mathematical language, he must make sure that his students really understand the meanings of the symbols used. As Leary says, "It is imperative that the student appreciate the value of appropriate symbols as a means of aiding and conserving thought, and of expressing thought succinctly. He must . . . make mathematical symbols an intrinsic part of his language and handle them as expertly as the words of his native tongue. If he reads *a* or *b* or any other literal symbol as a letter of the alphabet and not as a number with all the characteristics of any other number, or as a unit of value, or even as a part of speech, he is reading with neither meaning nor purpose beyond that of manipulating symbols."[24] The vocabulary building techniques described above in connection with new words and words having specialized meaning in mathematics may be used effectively in helping students develop mastery of the meanings inherent in mathematical symbolism.

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[18] William S. Gray, "Increasing the Basic Reading Competencies of Students," *Reading in the High School and College,* p. 97.

[19] Lancelot Hogben, *Mathematics for the Million,* New York, W. W. Norton and Co., 1937, chap. 3.

[20] Tobias Dantzig, *Number, The Language of Science,* New York, The Macmillan Company, 1930.

[21] Edward L. Thorndike, Margaret V. Cobb, Jacob S. Orleans, Percival M. Symonds, Elva Wald, and Ella Woodyard, *The Psychology of Algebra,* New York, The Macmillan Company, 1924, p. 89. Also quoted by Bernice E. Leary, *op. cit.,* p. 151.

[22] Leary, *op. cit.,* p. 151.

Growth in mastery of the factual content of mathematics will be concomitant with mastery of the vocabulary and symbolism of the subject. A teacher should always be on the lookout for references to ideas or concepts which may not be familiar to his students and seek to clarify all such allusions. With beginning classes, it is not safe to assume that references to even such well-known and fundamental concepts as the Pythagorean relationship have meaning to students.

Moreover, mastery of the technical vocabulary and symbolism of mathematics is not enough to insure competence in reading mathematical subject matter. Mathematical subject matter must be read intensively and many students have never learned to do this type of reading. As a consequence, Harris says, "They read mathematical problems casually, omitting data, overlooking key words, making wrong interpretations, failing to recognize even the most familiar terms or forgetting them immediately after reading, neglecting to

[23] *Ibid.* [24] *Ibid.*, pp. 152–153.

analyze and see relationships between parts, and generally missing essential meanings. The result is nothing short of disastrous. They are made virtually helpless by an unusual or novel problem, by superfluous or insufficient data, or by the necessity to scrutinize data for any considerable time before choosing and arranging them for solution."[25] Part of these difficulties may be ascribed to carelessness on the part of the student, but the principal source may be found in the fact that mathematical material is harder to read than most other content material and requires reading techniques different from those utilized for other subject-matter areas. Terry found that:

> The numerals of problems make greater demands upon the attention of readers than do the accompanying words, as is shown by the following facts: (a) the average number of digits included by a pause on numerals is decidedly smaller than the average number of letters included in a pause on words. (b) The average duration of pauses on numerals is greater than the average duration of pauses on words in the cases of all subjects. (c) The percentage of regressive pauses on numerals is greater than the percentage of such pauses on words. The explanation of the greater demand of the numerals probably lies in the fact that the combinations of digits in numerals are continually different, whereas combinations of letters in words remain stable.[26]

The mere presence of numerals and mathematical symbols is not the only factor which makes reading of mathematical subject matter difficult. The reader must be continually seeking to discover relationships, weighing meanings, and analyzing the material read to see what is implied, what conditions are presented, and what conclusions may be drawn. If it is a problem which is being read, the reader must pick out the essential data, determine what is required for a solution, and plan how to obtain that solution from the information given. Evidently a slow, intensive type of reading is required, and since students have usually had little experience with this type, they will need instruction and practice in reading techniques suitable to mathematical subject matter. Terry suggests that students should be taught to reread problem material as a means of gaining comprehension of the conditions of the problem. He recommends that:

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[25] Theodore L. Harris, "Making Reading an Effective Instrument of Learning in the Content Fields," *Reading in the High School and College*, pp. 159–160.

[26] Terry, *op. cit.*, p. 67.

> 1. Pupils should be taught to distinguish between the first reading and the re-reading phases of their attacks on problems.
> 2. They should learn to consider numerals and the accompanying descriptive conditions as different elements of a problem and separable for reading purposes.
> 3. During the first reading, they should devote their entire attention to the conditions of the problem.
> 4. At the same time skill should be developed in the partial reading of numerals.
> 5. While this skill is being acquired, pupils should be apprised of the essential similarity between the conditions of the problem and such details of the numerals as are perceived by partial reading.[27]

Terry considers a partial reading of a numeral to be any reading which does not take into account the identity and place of each individual digit in the numeral. Hence, his recommendations imply that on the first reading of a problem, or of expository material, only the most significant characteristics of numerals and symbols should be noted; a later reading, when the total situation is well in mind, should be reserved for the exact details of numerals and symbols. In reading mathematical material, students should be encouraged to use pencil and paper to write down formulas and equations and to carry out indicated operations. References to geometrical figures and to other explanatory material should be checked. In short, mathematical subject matter should be read slowly, intensively, and with precision. A reader should not expect to get the full implications of the ideas presented at the first reading but should reread passages as often as is necessary to master the ideas and concepts presented. Finally, reading mathematical subject matter is an active, reflection-level process. A reader cannot passively accept the author's results and conclusions. He must think through the relationships involved if he is really to understand the concepts and ideas presented.

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[27] *Ibid.*, p. 101.

A teacher of mathematics should be on the alert at all times for ambiguous or involved language in the materials presented to his class. He should provide explanations for all unfamiliar terms and symbols, references to little known theorems, and similar unusual demands upon the mathematical background and reading ability of his class.

This discussion has so far been confined to problems of reading textbook materials in mathematics and that largely at the level of secondary-school algebra. However, the difficulties of reading mathematical subject matter increase at more advanced levels because of the greater use of symbolism and the greater generality of the concepts considered. The suggestions given for developing increased power to read mathematical material at the secondary level will be found applicable to reading at advanced levels. And skill in effective reading of mathematical textbook materials at any level will find ready applications in reading scientific material, statistical reports, the financial pages of newspapers, and technical material of all kinds which requires careful, intensive reading. Not only should increased skill in reading mathematical materials be reflected in greater scholastic achievement in mathematics; it should help to make the mathematics learned more functional outside the mathematics classroom.

TEXTBOOKS AND TEACHING AIDS

Modern teaching theory has, in general, tended to emphasize use of multiple sources of factual material rather than dependence on a single textbook. However, probably because of the sequential and stable nature of mathematical subject matter, teachers of mathematics have continued to rely heavily on the single textbook. Is this practice undesirable in itself? What is the function of the textbook in teaching mathematics?

The function of the textbook in teaching mathematics is well expressed by the title of an early arithmetic—*The Schoolmaster's Assistant.* A good textbook provides an exposition of the subject matter of a course carefully organized with reference both to the logic of the subject matter and to the psychological considerations which make for effective learning. A textbook frees a teacher from the necessity of providing a collection of appropriate and well-graded exercises illustrating each topic and principle. It saves a pupil from the drudgery of taking class notes on the teacher's presentation of theory and from the errors which would result from incomplete or erroneous notes. The textbook will, in most instances, be the course of study. From these considerations, it seems evident that the textbook is a most important factor in the teaching process and that great care should be used in selecting the text.

The task of choosing a satisfactory textbook is complicated by the fact that there are so many to choose from, each with its claim to special excellence. Since educational theory has been in a state of turmoil in recent years with resultant confusion as to what is desirable in the way of content and method for the courses offered, the textbooks available mirror all facets of the swiftly changing theories as to what is or is not desirable in the way of subject-matter content for courses in secondary-school mathematics.

Suggested methods for choosing a textbook are little less numerous than the possible choices of textbooks. Perhaps the most promising is a score-card technique by which the most significant characteristics of a group of prospective textbooks are assigned numerical ratings. However, to work out an acceptable score card requires subjective judgments on the part of the score-card maker as to the relative importance of each characteristic listed, and use of the score card requires each teacher to rate the books judged in terms of his estimate of its merit with respect to the characteristics selected by the score-card maker. Hence, knowledge of the features which should characterize a good textbook is essential to the intelligent use of a scorecard or of any other method for choosing a textbook.

The following outline is suggestive of the features which need to be considered in choosing a textbook:

SUGGESTIONS FOR JUDGING TEXTBOOKS

I. Mechanics of the book.
  1. Is the binding substantial and attractive?
  2. Is the printing well done—good quality paper, type of proper size, page attractively arranged, etc.?
  3. Are the illustrations appropriate, well placed, and attractive?

II. Is the book suitable for use as a textbook in the course in question?
  1. Is it relatively recent?
    a. What is the date of the copyright?
    b. Is it a new book or a revision of an older edition?
  2. What are the qualifications of the author?
    a. What is his present position and what experience has he had which would qualify him to write a textbook?
    b. What are the author's qualifications to write a textbook from the standpoint of scholarship?
      (1) What degrees does he hold?
      (2) What other writing has he done?
      (3) What is his standing as a scholar?
  3. Does the subject matter content conform to accepted criteria?
    a. Does it conform to the recommendations in the Fifteenth Yearbook of the National Council of Teachers of Mathematics?
    b. Does it conform to the requirements of the state course of study?
    c. Does it conform to the syllabus of the College Entrance Examination Board?

III. Arrangement and presentation of material.
  1. Is the arrangement of the subject matter satisfactory?
    a. Is the logical organization of the subject matter adequately revealed?
    b. Is the order of presentation of topics and the style of writing appropriate to the maturity and the interests of the learner?
      (1) Are the terms and symbols used the ones accepted as standard?
      (2) Are the definitions given in clear and accurate terms?
      (3) Is the vocabulary suited to the age and grade of the student?

2. Is there suitable and sufficient practice material?
    a. Are the exercises well graded?
    b. Are the exercises significant problems or are they just "exercises"?
    c. What provision is made for review and maintenance of skills?
3. What provision is made for individual differences?

IV. What are the special features of this textbook?
V. How will it fit in to the local situation?

A mathematics textbook probably gets harder usage than any other textbook unless it be one in science. Therefore, the binding must be durable. It must be of a style which will permit the book to lie open on the desk. Nothing is more annoying than a mathematics textbook which will not lie flat. The type should be clear and easily read. If exponents, subscripts, and similar symbols are hard to read, unnecessary errors will result. It is desirable to have important definitions, formulas, and theorems in blackface type or emphasized in some similar way. Illustrations should be clear and appropriate. The current tendency to put into mathematics textbooks numerous pictures which contribute nothing to the content merely increases the cost of the book. The practice of using pictures of bridges, airplanes, dams, or other engineering structures is a case in point. The illustrations should contribute to understanding of the subject matter. There should be numerous diagrams and explanatory figures, but a picture of a bridge does not often add anything to understanding of a topic like quadratic equations. Pictures divorced from any relationship to the text are a needless expense.

Mathematics textbooks retain their youth better than do books in almost any other area but even in mathematics the changes in textbooks over a period of years are extensive and significant. A good textbook should not be discarded simply because it has a copyright date several years old. But, other things being equal, the newer of two otherwise equally desirable books is likely to be the better choice. A revision of a successful text will probably be a better choice than a completely new book. A revised edition should mean that the desirable features of the older edition have been retained, any errors eliminated, and any weak points strengthened.

Secondary-school mathematics textbook authors are usually either college professors of mathematics, college professors of education, high-school administrators, or high-school teachers of mathematics. From the standpoint of knowledge of subject matter, a college professor of mathematics is usually the best qualified of all these to write a textbook. But he may be unfamiliar with or unsympathetic to the problems and conditions peculiar to the secondary school. He is more likely to be primarily concerned with subject matter and to pay less heed to the needs of pupils than is one whose interests are less specialized. A college professor of education or a high-school administrator, if either has an adequate background in mathematics, may be able to write a satisfactory textbook. He will probably be familiar with the needs and problems of the secondary school but may lack the wide knowledge of the subject matter of mathematics needed to write a really good textbook. A high-school teacher of mathematics, if he has the requisite background of academic preparation and of practical teaching experience, may be well qualified to write an acceptable textbook. It is evident that the position held by the author of a textbook under consideration, his academic preparation, and his other writings are criteria that may be of some use in judging a textbook. In particular, such considerations afford a basis for judging the probable soundness of a new book which offers a marked departure from current practice.

Since the textbook does, to a large extent, determine the content of the course taught from it, it is most important that the book chosen conform in content and arrangement to the requirements of state courses of study and to other authoritative formulations of criteria for the choice of subject matter. In particular, in situations where significant numbers of the students will take the College Entrance Board examinations, the text must conform to the Board's syllabus.

The style and vocabulary of a textbook must be suitable to the age and experience of the students for whom it is written. If the symbols and terms are not those in common use, students will be confused and may fail to recognize the identity of two differing formulations. For example, college algebra teachers commonly find that students who are quite proficient in using the quadratic formula in solving equations of the form $ax^2 + bx + c = 0$ will fail to recognize that it may be used to solve $gt^2/2 + vt = s$ for t. Of course, in this particular instance the student should be able to recognize an equation as a quadratic and should realize that the formula will apply whatever the notation used. But the fact that students tend to be confused by even so apparently simple a matter as the transfer of the notion of the quadratic formula from one notation to another is an indication of the importance of having high-school texts conform in terminology and in symbolism to current usage.

If mathematics is to be a means of developing ability to think clearly and with precision, the textbooks studied should themselves be models of clarity and precision. All new terms should be clearly and accurately defined in terms comprehensible to the student. Possibly the most serious criticism that can be made of mathematics textbooks is that students cannot read them with understanding. Yet, too often, in an attempt to avoid the fault of being so precise and logical that the students cannot follow their exposition, authors go too far in the other direction and become so wordy that clarity is lost.

Mastery of mathematics can come only from solving problems which bring out the implications of the theory presented. Therefore, any satisfactory mathematics textbook will provide numerous sets of well-graded exercises illustrating the applications and implications of the theory presented. These exercises must be graded as to difficulty so that there are simple exercises at the beginning of a list which help to fix the basic theory; then, after the fundamental concepts are well established, should come numerous more difficult exercises which bring out special applications and unusual interpretations and which include some difficult enough to challenge the powers of the best students. Learning does not come most effectively until and unless a student is challenged and forced to exert his utmost powers to accomplish the lesson assigned. It is especially helpful to have several exercises grouped together, each being a variation of essentially the same problem. Then the teacher has ready at hand additional illustrative and practice material on any point or difficulty that may arise.

The problem of periodic review and maintenance of skills is always significant, and a good textbook will provide adequate material for this purpose. Provision for individual differences in the form of A, B, and C groups of exercises seems to be losing favor. However, additional lists of problems or other devices for providing further drill on fundamentals and of more difficult problems to challenge the better students are distinctly helpful.

Any special feature advanced as a particular virtue of a textbook should be carefully weighed in light of local needs and in light of the aims of the course. Sometimes an innovation becomes standard practice because it proves its value in practice. More often, an extreme innovation fails to catch on or is considerably modified before it receives widespread acceptance. It is a safe practice, unless the situation is definitely an experimental one, to adopt a policy of wait and see on extreme deviations from current practice.

There is available a bewildering array of workbooks, practice pads, and other teaching aids. Often such materials have considerable value as a means of providing for individual differences. Particularly if a class varies widely in ability, a good workbook may be of value in providing simpler materials for weaker students or more challenging problems for superior ones than are afforded by the textbook. Another type of workbook provides materials for periodic reviews and for semi-standardized tests over small blocks of subject matter. Properly used, a workbook may prove to be a valuable aid to a teacher. Unfortunately, too often it becomes merely a vehicle for the most uninspired sort of teaching. Too many workbooks were prepared under the theory that learning is habit formation and they provide repetitive drill to the point of nausea. A workbook may be a useful tool in teaching mathematics. It should not be a substitute for good teaching.

There is likewise coming to be available a wide assortment of mechanical aids to mathematics teaching. Motion pictures, slides, models, and instruments of varied types and uses are available. As yet, too few motion-picture films, either silent or sound, do anything which cannot be done as well by the teacher with chalk on the blackboard. The use of visual aids is certainly not new in mathematics. From the beginning, in its study, teachers have used diagrams and pictures to make their meaning clear and students have sketched figures to aid them in thinking through their problems. There is a distinct place for the motion picture in teaching mathematics but much remains to be done before the films available can be said to realize the full potentialities of the medium.

Wooden and plaster or string models are of great value in teaching three-dimensional geometry. Straightedges, compasses, and protractors are, of course, necessities in every mathematics classroom. Whether or not additional expensive equipment such as surveying equipment should be purchased will depend in large part on the experiences and teaching techniques used by the instructor. Some teachers make extensive use of field work in teaching mathematics. They will make good use of tapes, levels, transits, etc. Others, equally successful and competent, will not use such equipment sufficiently often to warrant its purchase.

PROBLEMS IN PUPIL ACCOUNTING

The first administrative function which a teacher must discharge—the first in point of time, that is, not importance—is that of keeping records of class attendance. The technique used in taking the roll will, of course, vary with administrative practices in individual schools. If, as is quite commonly the case, absence slips are collected each class period, the roll must be taken and an absence report made out at the beginning of a class period. If a report need not be made until the end of the day, the roll may be taken from daily papers handed in or in some other way which makes unnecessary a formal checking of the roll and permits the time saved to be used for more constructive purposes. It is sometimes possible to delegate responsibility for checking the roll to designated pupils. The device of requiring pupils to take assigned seats in the classroom permits a rapid check of absentees by merely noting the empty seats. Obviously, such devices as have been mentioned are not necessary in a small class but are possible solutions to what becomes a serious problem when classes are large. Whatever the technique used in checking and reporting attendance, a conscientious teacher will want to keep sufficiently detailed records to be able to make suitable make-up assignments and to assess the extent to which achievement has been affected by absences.

A second problem, largely administrative in nature but definitely part of the teaching process, has to do with keeping the records of pupil achievement on which term grades are based. The more adequate and complete these records are, the more reliable and useful will be the marks given by the teacher. But if a teacher attempts to keep detailed records of a large class, he soon finds that the sheer weight of the accounting task he has undertaken is draining off time and energy that could be used much more constructively. For example, there is a strong temptation to keep daily grades in mathematics classes since it seems a simple matter to assign problems to be worked out and handed in each day. A check on these daily papers, together with a record of such tests and formal examinations as may be given during a marking period, appears to make possible a fairly exact appraisal of the quality of work done by the pupil. Unfortunately, however, the task of collecting, checking, and recording marks on a large number of papers every day becomes a heavy burden on a teacher's time and energy. Many teachers are of the opinion that the conditions under which homework is done are so variable as to make it an entirely unreliable basis for marking. These teachers commonly give frequent tests and base their marks on the results. But, whatever the basis used for marking, the teacher must check papers, assign marks, record these marks, and report them periodically to administrators, and to pupils and parents.

Apparently pupil accounting—keeping records of achievement and attendance—is an important part of a teacher's work, but it is not his primary function. It is imperative, therefore, that each teacher work out a system for keeping such records as will be needed to make the type of reports required by the administration of the school in which he is working. The system used should be sufficiently complete to enable him to appraise and report accurately on each pupil's achievement but not so detailed as to constitute a severe drain on the energy he needs for teaching. The labor of correcting papers can be reduced if consistent use is made of the practice of having daily papers and short tests checked by the pupils in class. Use of objective or controlled-answer-type tests which can be checked quickly is another way to minimize the routine work of correcting papers. If daily papers are checked in class by the students under the supervision of the teacher or if short, frequent tests are given and checked in class, a fairly dependable basis for an estimate of pupil progress may be obtained by merely recording the number of exercises solved correctly by each pupil. These data may be quickly recorded and over a period of time give a reasonably accurate measure of pupil achievement, particularly when supplemented by occasional comprehensive formal tests, either teacher-made or standardized. The competent teacher will determine, by experiment, the type and extent of the records he needs to make the reports required of him and to make assignments for make-up work, for counseling purposes, and to appraise the effectiveness of his teaching procedures.

Routine management of the classroom affords numerous chances for a teacher to save time by careful planning of the work of a class. If the distribution of workbooks, test papers, and similar materials is planned in advance, much time and confusion will be saved. If the amount of shifting about by pupils during the period is kept to a minimum, there will be less confusion, less wasted effort, and less tendency to kill time. If the work of a period is so planned that every pupil is kept busy all the period, disciplinary difficulties will be minimized and pupils will be aided in developing worth-while habits of sustained industry and persistent effort. Teachers should always begin classes promptly on the hour and keep the work moving at a brisk pace throughout the period. Pupils become restless and inattentive when the room becomes hot and stuffy; hence attention to the proper regulation of ventilation and temperature of the room will pay dividends in terms of more efficient learning. In short, if a teacher will see to it that the atmosphere of the room is brisk and businesslike, if he maintains conditions conducive to good work, handles routine administrative procedures quickly and efficiently, he will have set the stage for an effective learning situation.

LESSON PLANS

To make sure that each day's work contributes toward the total objectives of the course, that all the materials needed are available when wanted, and that each portion of the work is given its proper emphasis, the teacher should make both an overall plan for the work of the term and also detailed plans for the work for each week and each day. There is a tendency for teachers of mathematics to assume that the textbook and the state course of study provide all that they need in the way of an overall plan. It is true that typical secondary-school courses in mathematics, with perhaps the exception of the course in general mathematics, are quite standardized, that reasonably satisfactory textbooks are available, and that because of the sequential nature of mathematical subject matter it is not usually wise to depart too far from the sequence presented in the textbook. But the amount of time to be spent on particular topics may vary depending on the ability and interests of a particular class. It is often necessary to order supplementary material well in advance of the time when it is to be used. Tests need to be scheduled with the dates of vacations and such disrupting factors as major athletic or social events in mind. For all such purposes, a long-range plan for the course is necessary.

Similarly, a teacher needs to make detailed plans for each week and for individual lessons. The weekly plan need not be elaborate; it need not be written down unless the administration requires something of the sort for the use of substitute teachers in emergencies; but it is essential that a teacher think through the work that is to be accomplished in the immediate future and that he have clearly in mind just what each day's lesson is to contribute toward that goal.

The daily lesson plan is, in turn, a careful analysis of the problem presented by the work of a particular day. In making a daily lesson plan a teacher makes his preparation for teaching the class. The plan to be followed may be written out; it may be merely a few notes jotted down to aid the memory; or it may be entirely in the mind of the teacher. Whatever physical form it takes, it should include a definite statement of the objectives of the lesson in terms of specific facts or techniques to be presented. In making his plan, a teacher should pick out the essential factors in the assignment, decide on the method of presentation, look up any additional examples or supplementary data he may wish to present, decide what pupil activities will be desirable and what the next assignment will be. In short, he should think through the problems he will meet in presenting the lesson to his class. The thoroughness with which he does this, the extent to which he anticipates and prepares for questions and difficulties, will determine to what extent the class session is a brisk, businesslike period in which effective learning takes place.

It is not uncommon for student teachers to become tired of writing the formal lesson plans of their student teaching days with their lists of teacher and pupil objectives, materials needed, procedures to be used, and so on, and to decide that they will not submit to such drudgery when freed from the compulsion of their supervisors. But good teaching cannot be extemporaneous. There must be planning in terms of available materials, the needs of the class, and the distractions of the moment in the form of extracurricular events, community interests, and world affairs. The material to be covered must be organized and a way of presenting it decided upon. It is not necessary that a teacher write out a formal plan, as is usually, and for good reasons, required of student teachers. But it is necessary that he consider in advance all the topics covered in the outlines for the lesson plans required of student teachers whether or not he writes down what he expects to do about them. Until a young teacher has had sufficient teaching experience to be sure he can keep in mind all the essential considerations during a class period despite the distractions that may occur, he should play safe and put his plans in writing. However, many good teachers do not write down elaborate plans. They do carefully think through just how they will conduct the class. They make plans even though they do not feel it is necessary to write them down. On the other hand, because no two classes are ever alike, a conscientious teacher will never simply follow last year's plan just to save himself the trouble of rethinking the lesson and making a new plan.

Making a lesson plan, whether written or not, is the process by which a teacher:

1. Formulates a definite statement of the contribution which this particular lesson is to make to the total objectives of the course.
2. Decides what subject matter will be utilized to achieve these objectives.
3. Determines what teaching materials will be used in presenting the chosen subject matter.
4. Determines the method to be used in presenting the lesson, including determination of what he will do for the pupils and what he will require them to do.
5. Prepares all the supplementary material needed.
6. Determines how to appraise the success of his teaching.
7. Decides what shall be done in the next assignment.

Making a plan is, therefore, an essential part of good teaching practice and should never be neglected.

THE ASSIGNMENT

Making the lesson plan is, of course, the preparation a teacher makes before teaching a lesson. But, if learning is to take place, the student must also do something. The tasks set pupils, either before a lesson is taught or as a consequence of the teaching of the lesson, constitute the assignment. Butler is of the opinion that the possibilities of the assignment as an important factor in the teaching process have been largely overlooked.[28] In the first place, he says, the possibilities of the assignment as a means of establishing objectives have not been realized. Obviously, a pupil can, and will, accomplish more if he knows exactly what he is supposed to do. As Butler points out, when an assignment is made merely in terms of so many problems or so many pages of subject matter, the assumption that the textbook will provide the pupil with an idea as to what he is to do and why is implied.[29] A second possibility inherent in the assignment which has been neglected, according to Butler, is that it affords an opportunity to motivate study of the material to be assigned. If the assignment is properly made, the pupil will develop an attitude of eagerness to know or do which is fundamental to learning.[30]

A good assignment will include giving adequate directions so that pupils can carry it out without further assistance. It will also make clear relationships existing between the material assigned and that which has been covered or which will be covered in the future. In short, an assignment should create in a pupil a desire to do the work required because he sees the reason it was assigned and how it fits into the larger aims of the course and his own purposes. It should also be sufficiently clear and explicit for him to know exactly what he is supposed to do and how to go about doing it.

The manner of making an assignment is also important. If it is to serve as a means of presenting objectives, of motivating study, if it is to provide explicit directions as to the procedure to be followed, the making of the assignment must not be deferred until after the ringing of the bell signifying the end of the period. It is such an integral part of the whole teaching process that sufficient time should be provided for the assignment in planning the lesson to achieve the possibilities Butler sees as inherent in it.

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[28] Frank A. Butler, *The Improvement of Teaching in the Secondary School*, Chicago, University of Chicago Press, 1939, pp. 169 ff.

[29] *Ibid.*, p. 170.

[30] *Ibid.*, pp. 172–175.

Young insists that it is not advisable to assign work until it has first been sufficiently developed in class for even the slowest pupil to do it with success and profit. He says, "The pupil should never be set to struggle with really new matter without the supervision of the teacher. The work assigned may require thinking, problems in some respect new may be given for solution, but there should always have been enough similar antecedent work to furnish pupils a clue sufficient to prevent their working in the dark."[31] Young considers that the most effective homework is that which completes the classwork of the day rather than that which prepares for the next day's lesson.[32]

In other words, the assignment should be a part of the total teaching process. Each lesson should be so organized that specific tasks are left for the pupil to do, either on his own or under supervision. These tasks should be clearly defined so that a pupil knows exactly what he is to do and how he is to do it, and the work assigned should have a definite and evident bearing on the work done in class. The exact form of an assignment will, of course, depend on the teaching method used. Unfortunately, too many teachers of mathematics have fallen into a pattern. At the beginning of a period they check on the results obtained by the pupils in their homework assigned the day before. Sometimes all the problems assigned are put on the board and explained, at other times only the most difficult ones. Then a new topic is presented and explained and an example or two worked out. An assignment is usually merely a matter

[31] J. W. A. Young, *The Teaching of Mathematics in the Elementary and the Secondary School*, New York, Longmans, Green and Co., 1925, p. 132.
[32] *Ibid.*

of designating certain problems to be worked for the next day. This procedure is used day after day without variation. The study of mathematics thus becomes a matter of working exercises—not problem solving in the sense of learning to think through a difficulty, but simply a matter of learning to use specific techniques for "answers" to specific exercises.

CREATING AND MAINTAINING INTEREST IN MATHEMATICS

One of the big problems of a secondary-school teacher of mathematics is that of creating and maintaining interest in the subject. Primarily, perhaps, because mathematics is usually associated with the logical and practical aspects of life, its recreational possibilities are seldom explored. Yet the average individual is quite unable to resist the allure of a mathematical puzzle. The origin of some of the puzzle problems that go the rounds today is lost in antiquity. The popularity of the puzzle section in youth magazines attests to the eagerness with which individuals respond to an intellectual challenge. There is, of course, little opportunity to bring such material into the work of a mathematics class from day to day. Whenever such opportunity offers, it should be taken. But the adolescent high-school student is fond of belonging to organizations and clubs. A mathematics club affords an excellent means of creating added interest in mathematics through emphasis on the recreational aspects of the subject. A considerable amount of material is available that is suited to the needs of such a club—materials which will both entertain and broaden a student's knowledge and understanding of mathematics. The actual form of the organization will depend on the time available for its meetings and other local conditions. It may be a club which meets at a free period in the day or it may be a group that is privileged to meet occasionally during part of a regular recitation period. Its program may be primarily social and recreational in nature or it may be a medium for exploring the applications of mathematics or for more extensive study of topics of interest to the group but not part of the regular course. Whatever form a mathematics club takes, it should be a student activity with the teacher as adviser and consultant only. It is not possible to enjoy the recreational aspects of mathematics by assignment any more than it is possible to "require democracy."

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FOR SECONDARY EDUCATION

Chapter 11

Evaluation of Instruction

BUSINESSMEN regularly take inventory—make an accounting of their gains and losses. Similarly, in school work, it is desirable to take periodic inventories of the educational gains and losses made by students. The instruments commonly used for this purpose are various types of pencil-and-paper tests. The process of appraising the results of tests given—the balancing of the account, as it were—is a matter of assigning grades or marks.

WHY GIVE MARKS?

Theoretically, school marks have three functions: administrative, instructional, and informational. Administrative functions are those relating to grade placement, promotion, graduation, and transfer from one school to another. The instructional function has to do with motivating students to greater effort and, presumably, to more complete realization of the objectives of the school. The informational functions are three in number: An assigned mark should indicate to a student and to others interested in his progress his instructor's appraisal of his achievement. It should afford his instructor, and the school administrator, a means of checking the effectiveness of the teaching being done. And it should provide a basis for predicting future achievement on the part of the student, both in school and out.

CRITICISMS OF CURRENT MARKING PRACTICES

Critics have attacked the value of school marks as administrative devices on the ground that they are too variable, unreliable, and inconsistent to provide a satisfactory basis for grade placement, promotion, or graduation. These critics insist that no one is sure just what a given mark recorded on a pupil's record means. There have been, and are, many systems of grading, and variations in the ways in which individual instructors assign marks within any system are so great as to justify to some extent the contention that school grades are not a satisfactory means of accomplishing the administrative functions of marking. It is common knowledge that some teachers are more demanding than others, that work which might receive a high mark in one class would be rated low in another. For example, Starch and Elliott, in a now historic investigation, found that the mark given the same paper by 115 different geometry teachers varied from 28 percent to 92 percent.[1] Later studies have added a voluminous literature on the unsatisfactory characteristics of school marks as criteria for grade placement, promotion, or graduation.

The instructional function of school marks has likewise been attacked. Wrinkle has characterized the use of school marks for purposes of motivation as "not only indefensible but pernicious."[2] His objections may be summarized as follows: (1) The motive for an activity should be associated with the activity itself; (2) school marks do not measure the socially significant aspects of education; (3) the idea that a student's mark is his pay for work done is an admission that the work done is meangingless, hence implies that revision of the curriculum is needed; (4) school marks create a competitive atmosphere and create or intensify unwarranted feelings of superiority on the part of naturally gifted students while adding to a sense of defeat and discouragement for less fortunate ones; (5) the system of comparison of one individual and his achievements with those of another is not a life situation; (6) owing to individual differences, varying environments, and previous educational experiences and achievements, there is no fair common basis for judging all students. Unquestionably, these criticisms of marking practices are, in many instances, warranted and therefore cast serious reflections on the effectiveness with which school marks, as currently administered, discharge their instructional functions.

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[1] Daniel Starch and Edward C. Elliott, "Variations in Teachers' Marks," *School Review*, 21:254–259, and 20:442–457 (September, 1912 and April, 1913).

[2] William L. Wrinkle, "School Marks, Why, What, and How?" *Educational Administration and Supervision*, 21:218–225 (March, 1935).

It is likewise obvious that, if school marks are as variable, unreliable, and inconsistent as has been charged, they are not dependable as a basis for information about pupil progress and quality of instruction. If the practice of giving school marks is, as Wrinkle claims, not only valueless but pernicious as a means of motivating students to do their best work, then it would appear that such marks constitute a most unreliable basis for judgments about vocational and educational aptitudes of students by parents, teachers, prospective employers, and the students themselves. In short, it would seem that a procedure with so many attendant evils and, apparently, so few virtues should be abandoned in favor of some scheme which promises to discharge more effectively the administrative, instructional, and informational functions of the traditional marking systems.

IN DEFENSE OF SCHOOL MARKS

But are school marking systems as completely nonfunctional as claimed? More important, even if present practices are ineffective, inconsistent, and unreliable, is there any way to obtain the information about a pupil's progress and aptitudes needed for informational, instructional, and administrative purposes other than through use of some kind of marking system? If the system—or systems—currently in use is not achieving the results it should, is not the remedy to overcome its weaknesses rather than to discard the whole concept of measuring *and* marking? There can be no doubt that sometimes students make attainment of high marks and not real learning their goal. Likewise, marks may prove to be incentives for cheating, hence potent breeders of dishonesty. There seems to be little room for doubt that the most effective learning situation is that in which the motive for learning is found in the activity itself.

However, the activity itself is only one possible source for motivating learning. There are others, even if they seem less desirable from a theoretical point of view. It is not difficult to imagine situations involving worth-while learnings in which the objective is far too distant to provide incentive for serious study by a pupil. For example, mastery of the fundamental concepts of zoölogy is distinctly worth while for a prospective medical student, yet its connection with his ultimate goal is too remote for that alone to serve to motivate his study of zoölogy. If the subject matter of zoölogy does not in itself stimulate his interest sufficiently to motivate adequate achievement in the subject, the necessity of making good marks in this required course if he is to continue his medical studies may be an effective, if theoretically less desirable, means of motivating him to attain the desired level of achievement.

The complaint that school marks measure only acquisition of skills and information is often true. The extent to which particular skills have been acquired and the degree of mastery attained relative to specific bits of information may be measured with a reasonable degree of objectivity and accuracy. From such objective measures a teacher estimates the degree to which students have achieved the less tangible objectives of the particular course and of the educational program in general. However, such procedure is in keeping with common practice in the world outside of school. One's associates constantly appraise one's achievements and compare them with their own. The estimates which an individual's employer or supervisor makes relative to the quality and quantity of his work are the basis for retention or dismissal, promotion, and pecuniary rewards. A teacher's judgment of a pupil's work has much the same character and is important and valuable for that reason.

Finally, the contention that school marks fail to perform the function of keeping teachers, pupils, parents, and prospective employers informed as to progress made, proficiency achieved, and aptitude revealed by students is often true. Yet it is highly desirable that this information be made available and any satisfactory marking system must fulfill this function adequately. A teacher needs information as to pupil progress in order to check the effectiveness of the teaching procedures employed. Administrative and supervisory officers need information of this character to determine what points in the total program of the school need special attention. Some means of determining when a student is to be promoted or graduated is essential. Information as to the progress made by individuals or groups is needed if their work is to be properly correlated in a coherent system. Pupils need information about their own achievements and potentialities. Parents want, and are entitled to, information about the progress, needs, and capabilities of their children. Prospective employers need some criterion for judging the abilities, attitudes, and work habits of the youth they are employing. In short, some form of school marking system seems to be essential.

WHAT CAN BE DONE ABOUT SCHOOL MARKS?

It appears, therefore, that school marks are not acceptable as a means of appraising and reporting pupil progress because they are inaccurate, unreliable, and subjective. Yet schools must have some means of obtaining, recording, and reporting appraisals of educational progress for administrative, instructional, and informational purposes. It also seems evident that most of the criticism of marking and marking practices is directed, not at the idea of measuring and reporting pupil progress, but at the way in which the measuring and reporting are done.

How can school marks be made truly descriptive of the educational achievement they purport to measure? What information about pupils and pupil progress is needed for administrative, instructional, and informational purposes?

School marks become more precise measures of pupil achievement as they become more objective in character. A mark is objective if no element of personal bias is allowed to enter into its determination, if the method used in determining the mark to be given is such that any competent judge given the same data about a particular student's achievement would reasonably be expected to give the same mark. By the same token, a mark is subjective if personal bias enters into its determination, if different markers would not agree closely on the mark to be given a particular bit of work.

Unfortunately, it is not possible to measure educational progress or aptitudes directly. Teacher judgments as to pupil achievement and aptitude must be based on indirect evidence. Proper grade placement, readiness for promotion or graduation, and probable ability to succeed in a given field of endeavor are commonly judged on the basis of scores made on examinations which in turn attempt to measure extent of mastery achieved over specific subject matter. Similarly, a student's aptitude for a particular study—or vocation—the extent of his previous knowledge of a given unit or subject, points at which his progress has not been satisfactory, and the probable reasons he has not made better progress in such instances are all inferred from scores made on various types of subject-matter tests.

However, a measure may be indirect and still be precise, objective, and reliable. For example, measurements of temperature are indirect, usually being inferred from the amount of expansion or contraction of a thin column of mercury or alcohol, yet no one questions their validity, accuracy, or usefulness. But such measurements are of changes which are direct consequences of changes in the attribute being measured—in this instance, temperature. Similarly, educational progress can be measured indirectly with precision and confidence if, and only if, changes in the attribute measured directly reflect progress toward the ultimate goals of education. Likewise, attitudes and aptitudes can be measured by pencil-and-paper examinations if the items included in the tests are such that failure or success on them directly reflects the extent to which an individual possesses or does not possess the trait we seek to measure.

As we have seen (see p. 123), the subject matter of a program of democratic education should be material which will (1) add to an individual's knowledge of the world, (2) help to harmonize his outlook on the world, and (3) stimulate him to think reflectively about his world and its problems. The success achieved by an individual or a class in such a program may be estimated from the extent to which (1) the information and skills deemed essential to further progress have been acquired, (2) acquisition of this information and these skills has resulted in greater competence to cope with and to understand the world, and (3) there is evidence of increased ability and willingness on the part of each individual to make reflective studies of his outlook seeking to discover ideas that are inadequate or in conflict and to resolve such inadequacies and conflicts. Complete evaluation of an individual's educational progress will include determination of (1) the probable degree of success which he may reasonably be expected to achieve in a given field, (2) the extent of his previous knowledge of a particular unit or subject, (3) the extent to which mastery of the information, skills, and attitudes expected to result from study of a particular course has been achieved, (4) points at which progress has not been satisfactory, and (5) diagnosis of probable causes of any apparent failure to make satisfactory progress.

The results of the evaluation must be recorded and reported to pupils, parents, administrators, and others entitled to receive such information—prospective employers, for example. But it is not enough merely to record and report scores made on tests and examinations. These scores must be interpreted in terms of their relationship to the purpose of the measurement. Just as a temperature of

98.6° F. is meaningless until we know that we are measuring a child's body temperature and that 98.6° is the normal or expected temperature, so a score of 90 on a test is meaningless until interpreted in terms of what it indicates in the way of achievement of educational objectives. It is the function of a marking system to provide an intelligible means of recording and reporting the results of a program of evaluating educational progress. But, since measurement must precede marking, let us turn first to a consideration of the instruments which may be used in measuring educational progress and discuss the problem of marking afterwards.

MEASURING INSTRUMENTS

If school marks are to be anything more than a subjective judgment on the part of teachers, measuring instruments of some form and description will have to be used. The measuring instruments used in school are, as we have seen, tests and examinations of various kinds. One convenient classification is to distinguish between intelligence tests and educational tests. According to one textbook on measurement in education, *educational* or *achievement tests* are designed to measure the results of instruction whereas *intelligence* or *psychological tests* are for the purpose of measuring the intelligence or mental ability of the pupil without reference to what he has learned either in or out of school.[3] Intelligence tests may be designed to measure either the general intelligence of a pupil or his ability in some special area or subject. In the latter case they are often called *aptitude tests.*

Educational tests may be used to measure general achievement in broad fields of knowledge or to measure specific attainment in relatively narrow subject-matter fields. The former type is often termed a *survey test,* and either type may be referred to as an *achievement test.* Such tests attempt to measure the extent to which a student has mastered the work of the grade or area or subject in which he is being tested. They are supposed to measure, as the name indicates, achievement rather than native ability.

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[3] Harry H. Greene, Albert N. Jorgensen, and J. Raymond Gerberich, *Measurement and Evaluation in the Secondary School,* New York, Longmans, Green and Co., 1943, p. 10.

Since not all students can profit equally from work in particular subject-matter fields and since it is undesirable to insist on students' taking work in areas for which they are unfitted by temperament, ability, and previous preparation, *prognostic tests* are sometimes used to predict the degree of success which a particular pupil may expect to achieve in a given subject. Prognostic tests differ from aptitude tests in that they are intended to measure a pupil's background skills as well as the extent of his endowment in the abilities essential to success in a given field. An aptitude test supposedly measures only potential abilities.

Diagnostic tests are intended to help a teacher discover the specific weaknesses and strengths of an individual student in a given area. It is assumed that the outcomes of the educative process in specific areas can be stated in terms of specific skills and understandings. The maker of a diagnostic test undertakes to identify and measure the extent of an individual's mastery of each such item.

Tests may also be classified as standardized or unstandardized. A test is said to be standardized when the items that make it up have been carefully selected in the light of accepted theories as to the most desirable objectives and subject-matter content of the particular course or unit, when the test items have been carefully arranged in order of uniformly increasing difficulty, and when norms have been established by actual giving of the test to large numbers of supposedly typical students for that course and grade. These norms make it possible to compare the achievement of an individual or group with that of other individuals and groups and so to judge relative achievement levels.

Most teacher-made tests are, of course, unstandardized. Since the techniques of standardization are relatively new—at least in comparison to the long history of mathematics—and since they are almost universally of the objective or "controlled-answer" type[4]

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[4] See Ernest E. Bayles, *The Theory and Practice of Teaching*, New York,

(teacher-made tests have been traditionally essay-type examinations), both intelligence tests and controlled-answer tests are commonly thought to be recent innovations. The controlled-answer form of test may be new but the idea of aptitude and intelligence tests is not. Gideon chose his soldiers by the manner in which they drank from a brook—a crude test, to be sure, but none the less an aptitude test. Plato discussed the use of aptitude tests in his *Republic,* and history is full of allusions to the various tests by which men have sought to measure aptitude or to predict probabilities of success in given undertakings. The contribution of modern times is simply greater precision and persistence in applying these concepts.

The most accurate method of testing aptitude is the test of life itself. But this is too costly in time, wasted opportunities, and heartbreak. Society has long sought simple means by which the presence of special aptitudes could be detected. In this connection school grades were held to indicate something of what might be expected of an individual in his after-school career. Various other schemes for predicting aptitudes such as personal appearance (physiognomy), conformation of the head (phrenology), or shape of the hands have had their advocates, but most of them have been found lacking in validity or reliability or both.

USES OF TESTS

The work of Binet and the other pioneers in the field of psychological testing resulted in development of reasonably dependable aptitude and general-intelligence tests. Similarly, a profusion of diagnostic, survey, and prognostic tests have been developed, published, and offered for sale. However, any published test must have wide applicability if it is to be sold in profitable quantities. Such tests are usually better adapted to measuring general achievement than achievement in specific portions of work done by a particular class.

Harper & Brothers, 1950, p. 315. Bayles suggests, and rightly, that the term *controlled answer* is more accurately descriptive of the so-called "new type" examination than the more commonly used term *objective tests.*

These tests, when properly standardized, and when state-wide and nation-wide norms are available, provide an excellent means for comparing the work of a particular class with the performance of students of comparable age and grade elsewhere. They are perhaps useful for determining proper placement of transfer students, for determining extent of the mastery achieved by a class over large blocks of subject matter, for predictive ends, and for diagnostic purposes. In short, ready-made, published, standardized tests have a definite place in a testing program which is an integral part of the total teaching process.

Tests prepared by a teacher, with the needs of his particular class and the particular objectives he is striving to reach in mind, have a flexibility and adaptability not possible with ready-made tests. They can be made to cover any particular block of work, to fit any desired time schedule, to emphasize any particular topic or item the teacher desires. So, for practical purposes, the most frequently used and significant tests are those prepared by a classroom teacher for use with his own classes.

Teacher-made tests may be used to check the efficiency of teaching. If students have achieved the objectives set up as desirable and can demonstrate their mastery of techniques and their command of facts and concepts on a test, a teacher has evidence that the procedures he has employed were effective. If, however, a large proportion of the class does poor work on the test, he knows that his teaching was for some reason not effective. A major function of tests, and especially of teacher-made tests, is, therefore, to check on the effectiveness of the teaching which has been done.

Secondly, teacher-made tests serve to show a student the extent to which he is accomplishing the tasks set him. Often a student honestly feels that he is making an adequate effort to master a course, that he does understand the material being presented. An adequate test over the material will either confirm his belief or show him the extent to which he is mistaken. A teacher-made test, because it specifically applies to the material the class has been studying, is a valuable means for this type of self-appraisal by pupils. Providing an incentive for study and a means of measuring adequacy and effectiveness of a student's study habits is therefore another important function of teacher-made tests.

A classroom test should always be considered as a teaching device. The preparation made by pupils previous to the test will afford them an opportunity to organize and relate topics which is not so readily possible in working on a daily assignment because of its more limited scope. Merely putting items into a test serves to emphasize these items. A student is chagrined at missing an item on a test and, particularly if the test is discussed soon after he has taken it, will be motivated to have any misconceptions or misunderstandings cleared up. For a teacher, the results of a test will be of great value for diagnosis or guidance. Careful analysis of the errors made on tests will indicate what needs to be done in the way of remedial work or reteaching. Pupil performance on tests provides criteria for use in counseling students as to study habits and about vocational and educational objectives.

The most common use of tests—that of providing a basis for marking—is really a minor and secondary function. If marking were the only function served by tests, the time required to make, give, and score a test and to assign marks based on the scores would be far too great to justify giving one. But if each test is used to the fullest extent as a means of evaluating achievement, both by pupil and teacher, if it is used to stimulate study, to emphasize important facts, ideas, and relationships, for diagnosis of difficulties, and for guidance as well as marking, it is an important part of the whole teaching process.

TYPES OF TEST USEFUL IN MATHEMATICS CLASSES

A teacher, when he comes to make out a test, must choose what type of test he will give. Traditionally, a mathematics test was a matter of solving problems, or in geometry of proving theorems or making constructions. If a student could solve the problems, prove the theorems, or make the required constructions, it was assumed that he had achieved the objectives of the course. Such a test was, of course, comparable to an essay-type examination in other fields and was the type on which Starch and Elliott based their study of the variation in teachers' marks mentioned previously.

A traditional, problem-type test in mathematics is open to some serious objections. If an instructor, intentionally or otherwise, selects problems in which the solution depends upon some unusual trick or device or if the problems are obscurely worded, the test may prove to be a measure of persistence and ingenuity rather than one of mastery of the concepts and techniques studied. A test is also likely to be an unsatisfactory measure of performance or understanding if the problems given are not graded as to difficulty and if the time element is not given careful consideration.

One common misconception relative to tests in mathematics is that they are easy to grade—"a problem is either right or wrong." But the error which resulted in the wrong answer may be minor—a dropped sign or an exponent omitted or miscopied—or it may be due to a complete lack of knowledge or understanding of the technique or principle involved. If no credit is given for any problem missed, minor errors are penalized as heavily as major ones; a student who has mastered the material and understands the principles but is careless may get a lower grade than one who actually knows far less about the material but who makes no errors on the few problems he is able to work. If part credit is given, however, subjective evaluation enters in, and the wide variations in teachers' marks reported by Starch and Elliott and others become possible.

Because of these obvious defects in the traditional type of test in mathematics, the "controlled-answer" or "objective-type" test is favored by many teachers. In mathematics, it takes the form of many simple problems or items, each of which tests a student's grasp of one detail of a process or idea, instead of a few exercises which attempt to measure his grasp of the concept as a whole. For example, in a geometry test each step in the proof of a theorem is made an item in the test. Thus there may be ten or more items on this particular theorem, in each of which a student is asked to write a word or phrase in a blank or to choose among several suggested answers. Usually the questions are so framed that the correct answer may be given to any one without reference to whether the student was able to answer preceding questions correctly. Obviously, such an examination would permit a student who knew something about the general procedure for proving the theorem to be given credit for the items he was able to mark correctly whereas, on the old-style examination, if he was unable to complete the proof of the theorem, he might be given no credit for a relatively large portion of the test.

An objective or controlled-answer type of examination is easier to take and to score than is a traditional type. Since a student need only write a letter or word or phrase as the answer to each item, the factor of sheer physical fatigue, which is sometimes a serious one in long, traditional-type examinations, is not so likely to become operative. Since answers are brief and specific, scoring can be done rapidly, even mechanically. And, since each item requires less time, more items can be included; hence a wider sampling of a student's knowledge of the material is obtained. Because of the more objective scoring, there will be less variation due to differences on the part of persons scoring the tests. Therefore, scores recorded by different teachers using the same test or by the same teacher for different classes can be more readily compared. It is easier to prepare alternate forms and so keep down cheating and to provide for retesting and for make-up tests.

On the other hand, a controlled-answer test is much more difficult to prepare. For this reason, there is a tendency to use the same test over and over again. This procedure may be desirable from the standpoint of comparing the work of one section with that of another but it has the less desirable consequence of tending to "freeze" the subject matter of a course to the specific material covered by the test. Moreover, a teacher will, consciously or unconsciously, teach for the test rather than for the real objectives of the course.

Because a student knows that the same test is likely to be used again and again, a premium is put on obtaining copies of the test in advance, and the testing procedure becomes, not a learning process, but a battle of wits between the teacher and the class. Obviously, the problems growing out of using the same test over and over and the problem of cheating exist with the more traditional problem-type tests also, but the controlled-answer test is especially vulnerable in this respect.

Moreover, since memory of facts is more readily tested by the controlled-answer type of test than ability to reason, unless the teacher is unusually careful the testing becomes a process of checking factual information only. With little or no opportunity for organization, for selection of data, for reflective thinking on the part of students, tests tend to become merely measures of memory. The possibility of a lucky guess is always present in controlled-answer tests. An intelligent student with a hazy knowledge of the material covered is often able to eliminate possibilities down to the point where he is able to guess a sufficient number of correct answers to make a good score even though he possesses in reality only a mediocre grasp of the subject. And, most serious of all objections to the controlled-answer test, it does not test the ability of a student to apply the material of the course in the way in which he will use it out of school. In life we are not often presented with several possible answers and asked to determine which of them is correct. We must take the data available and determine from them what appears to be a satisfactory solution to the problem facing us. The traditional-type examination lends itself somewhat more readily to checking a student's ability to do the type of problem solving met in life situations than does the controlled-answer test.

In short, both the traditional-type test and the newer, controlled-answer or objective tests have advantages and serious disadvantages. Both should be used on occasion. Carefully prepared, administered, scored, and marked, either may be an effective measuring instrument and teaching device. Carelessly and hastily prepared, carelessly administered, scored, and marked, neither type is worth the time and effort expended by pupils and teacher.

A review of the steps involved in giving a test will serve to show that neither type of test is free from a large element of subjectivity. For example, in giving a test, the following decisions must be made by a teacher:

1. Whether to give a test and the time to give it must be decided upon.
2. The objectives to be achieved by the test must be determined.
3. A decision as to the scope of the test—what material is to be covered—must be made.
4. The type of test to be given must be settled—whether it is to be a problem-type test or a controlled-answer type. If the latter, it may be a completion type, multiple-choice type, matching, true-false, or a combination of these kinds of controlled-answer tests. It may even involve an essay-type section.
5. There must be a decision as to the items to be included and the questions formulated.
6. The questions prepared must be arranged in some order.
7. A decision as to the length of the test must be made. This will, of course, be determined in part by the reasons for giving the test, the time available, and similar considerations.
8. The conditions for administering the test must be determined. For instance, is it to be duplicated; will there be more than one form; will any special precautions against cheating such as alternate seating be taken; will the time allowed to complete the test be limited; what use of notes and the text will be permitted?
9. After the test has been given, it must be scored. This step is, of course, "objective" in the controlled-answer type of test. It may be made relatively so in the older type.
10. The scores must be evaluated and marks assigned.
11. Finally, if the test is to be a teaching device, the teaching procedures to be used in light of the results obtained must be determined.

Evidently, of all these steps and decisions, only the scoring may be "objective." All the others involve subjective judgments on the part of a teacher. And they should be subjective. After all, if testing is a part of the teaching process, it too will be determined in part by the personality and methods of a teacher in light of the class situation quite as much as will his other teaching procedures. The criticisms of testing and marking practices are not, in reality, criticisms of their subjective character. They are criticisms of their lack of consistency, of their arbitrariness. And since neither type of test can automatically prevent arbitrary and inconsistent judgments by a teacher, the problem becomes one of suggesting procedures by which the undesirable features of either type may be reduced to a minimum and the values to be gained realized to the fullest extent.

TESTING TECHNIQUES

The details of test construction, administration, and evaluation are, of course, too extensive to be considered here. Certain basic concepts and techniques can, however, be considered with propriety and profit. In the first place, since testing—evaluation—is an integral part of the teaching process, tests and testing must always be studied in the light of the educational objectives sought. If the aim of the teaching process is to impart factual information or skill in certain techniques, the tests used should determine whether this information has, in fact, been acquired or the requisite skills developed. If, on the contrary, the aim of instruction is development of certain attitudes of mind or habits of thinking, a testing program should aim at determining whether or not the approved attitudes have developed, whether or not the desired habits of thinking have been acquired. And it is not safe to assume that measurements of the factual information acquired in geometry, for example, may be taken as evidence that the habits of thinking exemplified by that subject have been acquired along with the information.

The first step in preparing a test is, therefore, to state the objectives of the course clearly in terms which admit of evaluation. This is not always easy. For example, the word *appreciation* is frequently used in stating educational objectives. Thus a state course of study lists as an objective for study of ninth-grade algebra: "Develop in pupils an appreciation and understanding of the language of algebra." How can this objective be translated into terms that can be measured by a classroom test? Many pupils come into an algebra class with a vivid "appreciation" of the "facts" that algebra is "hard," "useless," and of value only as a college-preparatory subject. That is, *appreciation* is only another word for an attitude toward a subject. How, then, can a student develop an "appreciation" for the language of algebra? If he learns to use the algebraic language effectively, he may as a consequence come to "appreciate"—that is, to realize—its utility, its generality, and its simplicity. If his mastery of its use is limited, his "appreciation" of its value is correspondingly limited. If we state the objectives of our teaching of algebra in terms of concepts to be mastered, or skills to be acquired, we can measure with reasonable objectivity the extent to which these goals have been achieved. If we find that a pupil can actually use the techniques of algebra to formulate statements of quantitative relationships, if he can translate verbal problems into equations, solve these equations correctly, and verify his solutions by showing that the conditions of the problem have been met, we may well consider that he does have an "appreciation of the language of algebra."

MAKING TESTS

Once we have stated our objectives in terms which permit testing to determine whether they have been achieved, the next problem is that of formulating test items which will serve to demonstrate whether or not the desired learning has taken place. What should be the nature of the test? The traditional mathematics test was a matter of problems to be solved. Is such a test an adequate and satisfactory means of measuring understanding of principles? Does it put undue emphasis on technical skill? Does ability to solve selected problems successfully measure growth in ability to do critical thinking in nonmathematical situations? If a student fails to solve a problem correctly, what does his failure indicate? If a test made up of problems to be solved is comparable to the essay-type examination in other fields, is it possible to construct mathematics test items comparable to those used in controlled-answer tests in other fields? That is, is there any place for true-false, matching, multiple-choice, or completion-type test items in mathematics tests?

Objective or controlled-answer items can be used effectively to measure understanding of vocabulary, factual information about processes and relationships, and details of problem solving in mathematics. For example, suppose we wish to check the extent to which pupils in a geometry class have mastered the vocabulary of a given unit. We might, of course, give a test consisting of items of the form:

1. Define: parallelogram; trapezoid; rhombus.
2. What term is applied to the point at which: (a) the medians of a triangle meet; (b) the angle bisectors of a triangle meet; (c) the altitudes of a triangle are concurrent?

Or we may use the controlled-answer technique of matching items and formulate a sequence such as the following:

DIRECTIONS: Write in the blank in front of each statement in Column I the letter indicating the term in Column II described by that statement.

Column I

..... 1. A quadrilateral having both pairs of opposite sides parallel.
..... 2. The point of intersection of the medians of a triangle.
..... 3. A quadrilateral with opposite sides parallel and adjacent sides equal.
..... 4. The point of intersection of the altitudes of a triangle.
..... 5. A quadrilateral having one pair of sides parallel.

Column II

a. quadrilateral
b. trapezoid
c. incenter
d. centroid
e. rhombus
f. rectangle
g. parallelogram
h. orthocenter
i. circumcenter

If the matching type of test is to be used, care should be taken that the number of items in Column II exceeds the number in Column I and that the terms listed in Column II are a complete listing of the possible items in this unit of work. If this precaution is not taken, the test may turn out to be a check on students' skill in eliminating possibilities rather than a check on their knowledge of the vocabulary of the unit.

The true-false type of test item may sometimes be used to advantage to test for factual information, knowledge of the details of geometric proofs, or understanding of fundamental processes. Thus, in algebra we may wish to check students' abilities to recognize type forms in factoring. We may, of course, test them by items such as the following:

> Find the prime factors of:
>
> 1. $4x^2 - 9y^2$
> 2. $x^6 - y^6$
> 3. $2x^2 - x - 15$

Or we may put the same items into the form:

> DIRECTIONS: In each of the following exercises mark a T in the blank if the suggested solution is correct and an F if it is incorrect.
>
> ..... 1. $4x^2 - 9y^2 = (2x - 3y)\ (2x + 3y)$
> ..... 2. $x^6 - y^6 = (x^3 - y^3)\ (x^3 + y^3)$
> ..... 3. $2x^2 - x - 15 = (2x + 5)\ (x - 3)$

In geometry, true-false items are often used to advantage to check understanding of the statements of theorems. Thus, instead of a test item in the form:

> If two angles in a plane have their sides respectively parallel, what two possible relationships may exist between the angles?

we may wish to ask the same question in the following form:

> GIVEN: $\angle ABC$ and $\angle XYZ$ in a plane have $AB \parallel XY$ and $BC \parallel YZ$.
> ..... 1. $\angle ABC$ may be equal to $\angle XYZ$.

..... 2. $\angle ABC$ may be the complement of $\angle XYZ$.
..... 3. $\angle ABC$ may be the supplement of $\angle XYZ$.

and instruct students to mark the blanks T or F.

Completion-type tests may be used to advantage in testing for grasp of details in problem solving in algebra or in formal proofs in geometry. For example, suppose we wish to test ability to analyze reasoning problems in algebra. We may give a test composed wholly or in part of items such as:

> A grocer wishes to form a mixture of 60 pounds of coffee worth 80¢ per pound by mixing coffee worth 60¢ per pound with coffee worth $1.10 per pound. How many pounds of each kind of coffee should he use?

We may state the problem and infer the extent of a student's ability to analyze any problem from his success in solving this particular one. Or we may follow the above statement of the problem by a series of completion exercises which are designed to enable us to check more precisely the reasoning used by the pupil in solving the problem:

1. We should let x be ....
2. Then y will be ....
3. The 60 pounds of the mixture will be worth ....?
4. The .... pounds of $1.10 coffee are worth ....?
5. The .... pounds of 60¢ coffee are worth ....?

This same device may often be used to advantage in tests on formal proofs in geometry. Thus a test item may be of the form:

> Give a formal proof of the theorem: Base angles of isosceles triangles are equal.

Or we may use the proof of the same theorem as the basis for a series of test items of the form:

THEOREM: *The base angles of an isosceles triangle are equal.*
GIVEN: $\triangle ABC$ with side $AB =$ side BC.
1. TO PROVE: ....
CONSTRUCTION: BD bisects $\angle B$ and meets AC in D.
PROOF:

Statement	Reason
2. $\angle ABD = \angle CBD$	....
3. $\triangle ABD$	Two triangles having two sides and the included angle of one equal respectively to the corresponding parts of the other are congruent.
4. $\angle BAD = \angle BCD$	....

It seems evident that, in either instance, use of completion-type items affords opportunity to check mastery of specific details more readily and precisely than is possible with the more traditional type of examination. Because more direction is given to a student's thinking as he takes the test, he will need to spend less time seeking clues to a solution of any particular item; hence a greater number of items may be included on any particular test.

The multiple-choice type of test item may also be used to advantage in mathematics tests. For example, instead of the matching technique to check understanding of vocabulary, we may use items of the form:

..... The point at which the angle bisectors of a triangle are concurrent is known as:

a. The circumcenter; b. The orthocenter; c. The incenter; d. The centroid; e. None of these.

and instruct the pupil to write the letter indicating the correct answer in the blank. We may also use multiple-choice items to test skill in specific techniques. Thus, instead of asking a student to factor the binomial $x^6 - y^6$, we may present him with an item in the following form:

..... The prime factors of $x^6 - y^6$ are:

a. $(x^3 - y^3)\ (x^3 + y^3)$
b. $(x^2 - y^2)\ (x^4 + x^2y^2 + y^4)$
c. $(x - y)\ (x + y)\ (x^2 - y^2)\ (x^2 + y^2)$
d. $(x - y)\ (x + y)\ (x^2 + xy + y^2)\ (x^2 - xy + y^2)$
e. None of these.

In using the multiple-choice type of question, care should be taken to make the suggested answers reasonable. That is, they should be the answers a student will get if he makes the more common types of errors. Thus, in the factoring example above, students are likely to forget that, after factoring $x^6 - y^6$ as the difference of two squares, the factors so obtained can be factored still further. Or they may see the problem as one of factoring the difference of two cubes and again fail to reduce the result to prime factors. Analysis of the suggested answers will reveal that each of these more common errors is represented. The "None of these" option will serve to catch the individual who makes some unforeseen error. However, if the "None of these" item is to be used, there should be an occasional question on the test for which no correct answer is given.

These examples will serve to illustrate how various types of questions may be used in mathematics tests. As the discussion of each implies, no one type of exercise is superior in all situations and for all purposes. Nor should the teacher come to the conclusion that the controlled-answer type is always superior to the more traditional problem tests. Each type has its virtues and its faults. The teaching situation and the preference of the teacher will largely determine which will be used in a particular instance. Because of the greater objectivity in grading, standardized tests are almost always of the controlled-answer type. Because of the greater amount of time needed to prepare a controlled-answer test, teacher-made tests are likely to be predominantly of the traditional problem type.

Obviously, whatever type of test item is used, the questions should be clear and specific. "Trick" questions or problems in which the solution depends on some unusual interpretation of a word or phrase or on some unusual application of a technique are not usually valid examination questions. In so far as is possible, the questions on an examination should be graded as to difficulty. There should be some items which all students may reasonably be expected to solve correctly and a few items sufficiently difficult to challenge the most able student. It is a rather widely accepted axiom of test construction that a test on which a pupil makes a zero score has failed to measure how little he knows or has accomplished while one on which a student makes a perfect score has failed to measure the extent of his mastery of the material covered by that test. Obviously, there are exceptions to this rule. If the objective of the test is to determine whether a given degree of mechanical skill has been acquired, the test must be passed by all students and we are not interested in knowing by how much any given individual exceeds this minimum. If we are trying to find whether a group of pupils or an individual knows certain things which are taken to be prerequisite to something else we wish to teach, we are not so much interested in how much less than this required minimum the pupil knows as in establishing that he does or does not possess the specific facts or skills in question. For example, a zero score on a pretest simply indicates that a student has no previous knowledge of the material in question. Tests such as those given to aspiring actuaries or public accountants are aimed at determining if the individual tested does have the requisite mastery of the field as indicated by his score on a formal test. Failure simply indicates that as yet he has not achieved his goal.

A classroom test, however, will usually show a spread of achievement over a considerable range on the part of pupils taking it. In preparing a test, therefore, one should include items calculated to show the status of each member of the class with respect to the objectives of the course. Not only should there be some items which are relatively easy and some which are relatively difficult but there should be some items in which specific principles or techniques occur alone and others in which several ideas occur in combination. Such items will enable a teacher to determine if a student has mastered the basic concepts and at what stage of complexity in their application he has difficulty.

In some instances, of course, the major objective of an examination will be to test for mastery of specific facts or skills. Performance on a test in which pupils are asked to solve exercises involving these facts or skills will usually provide a reasonably satisfactory measure of achievement in such cases. But more often in secondary mathematics the objective is to determine whether a student can actually use the concepts and skills studied in the solution of typical problem exercises. If his error on a test is one of fact, we often cannot determine whether he would have solved the problem correctly had the correct facts been available to him.

The test of effective teaching is, of course, actual performance after leaving school. Hence, a testing situation should reproduce, in so far as is possible, out-of-school demands on the student. For example, except in routine work, an adult does not depend on his memory for specific facts needed in the solution of a problem. Instead, he ought to know where to find the information needed, and the crucial factor is whether or not he can select from all the data available that which is pertinent to his situation and use it to solve the problem at hand. In classroom testing, a comparable situation would obtain if students were given ready access to textbooks or such handbooks as they wished to bring to class. The obvious conclusion to this line of reasoning is that an open-book examination is the type which most nearly approximates out-of-school conditions. A little experimenting will soon convince a teacher that an open-book examination can, in fact, be most searching. A student commonly greets the first announcement of an open-book examination with the feeling that it will be easy. After experience with a well-made open-book test, the common reaction is that it is difficult, although still preferable. When he is freed from the necessity of memorizing masses of data which he fears he will need later, a conscientious student is able to devote his time and energy to systematizing and organizing his ideas about the material. As a consequence, he comes to see the relationships which exist in the subject matter studied and to see the course as a whole, with a definite gain in understanding and power.

AWARDING MARKS

The major problem arising out of a testing exercise is not that of making, administering, or even scoring tests. What usually proves most vexing to a teacher is properly marking a student's paper after it has been scored. Most of the criticisms of testing and marking practices are, in reality, not criticisms of testing procedures as such or of school marks, but criticisms of the way in which the marking was done. Traditionally, marking was on a percentage basis. Presumably, 100 percent represented perfection, and the actual grade indicated the amount by which a pupil had failed to achieve perfection. Such a system was open to numerous attacks. Obviously the whole scale was not used. Commonly some arbitrary figure such as 60 percent or 75 percent was taken as the "passing mark" (i.e., any score below this mark was judged to be indicative of work of such poor quality that no credit could be given). In effect, therefore, "passing work" was ranked on a scale which might vary from twenty-five to forty points instead of the supposed 100-point scale. Moreover, there was actually a greater difference between a grade of 95 percent and one of 97 percent, for example, than there was between a grade of 75 percent and one of 77 percent. Emphasis was on the errors made, not on actual achievement, and the scale used implied a greater degree of accuracy in marking than was actually achieved. Apparently teachers had evaluated a student's work with an accuracy of 1 percent or less. Actually, an instructor could seldom differentiate between the work of two students with anything like that degree of precision.

To avoid the manifest difficulties involved in use of a percentile system of marking, the scheme of using letters or other symbols on a four- or five-point scale was evolved. Sometimes as few as two categories are used, as when a student is merely marked "passed" or "failed," S or U, or some variation of these two marks. The most common practice is to use a five-point scale such as A, B, C, D, and F. On it a mark of A represents high achievement; a mark of B better than average work; a C average achievement; a D work that is below average in quality but still acceptable; and an F work so deficient in quality, or quantity, or both, that credit cannot be given. However, in using such a scale, even though it avoids the criticisms leveled at either the percentage scale or the "pass or fail" system, a teacher is still faced with the problem of determining how to translate the scores of tests, daily papers, and other evaluations of a student's work into marks.

As we have seen, some type of marking system seems to be necessary for administrative and informational purposes. It appears that, properly used, a five-point scale is reasonably satisfactory from a practical point of view. Probably the most valid criticisms of school marks grow out of misuse of a system rather than out of faults of the system itself.

How, then, can we assign marks to the scores made on tests so that the informational, instructional, and administrative functions of the mark may be discharged and still avoid the variation, inconsistency, and personal bias which so often make such marks of doubtful validity?

It is well known that individuals differ greatly in academic ability, in willingness to study, and in numerous other traits which affect scholastic marks. Experience has shown that, whatever the basis for marking used, there will be a relatively large group whose performance is "average" or undistinguished by either unusual merit or notable lack of merit. About equal numbers will achieve notably above average and notably below average. There will be a few whose achievement is definitely superior and a few whose work is so unsatisfactory as to be totally unacceptable. Similarly, in all types of measurements, the values found by measuring the individuals in a population tend to cluster about a mean with deviations from this mean decreasing in frequency as the size of the deviation increases. Distributions which present these characteristics are termed "normal distributions" from the so-called "normal curve of error" developed by De Moivre and Gauss. Unfortunately, however, the assumptions on which the normal curve is predicated and which should therefore be approximately attained if the curve is to be followed closely simply do not describe an ordinary classroom situation. The normal curve applies only to random samples selected from an infinite population. It gives dependable results only in situations in which large numbers of measurements are involved and in measurement of traits which are symmetrically distributed about the mean of a large population. A distribution of the grades given in a school system by many teachers to a large number of pupils will often be found to approximate a normal distribution but the average class is too small for the curve to apply exactly. Again, if the individual teacher finds that his marks deviate too much from a normal distribution over a period of time, he should study his marking habits and his classes to see if he is letting some personal bias affect his marks.

With the limitations and characteristics of the normal curve in mind, it is not difficult to see how uncritical recognition of the simple fact that school marks, when large numbers of marks were considered, were found to approximate a normal distribution led to the conclusion that the problem of assigning marks might be solved "scientifically" by the method of fitting a normal curve to the data by use of appropriate statistical procedures. Failure on the part of administrators and teachers to recognize that a small class might not be a random sample of an infinite population led to unintelligent insistence on uniform numbers of A's and F's, and of B's and D's. Many criticisms of the practice of giving school marks had their origin in such obvious misinterpretations of the theory of the normal curve as the idea that in every class every A grade must be balanced by an F grade and that a fixed percentage of every class must be assigned to each letter grade group.

However, there is a device based on normal curve relationships which beginning teachers often find helpful in solving the perplexing problem of awarding grades fairly. If, after a set of papers has been scored, a frequency distribution of the scores is made and the mean and standard deviation of the distribution calculated, letter grades may be assigned as follows: (1) Divide the standard deviation (expressed in terms of the original scores) by two and add the result to the mean score. This value may be taken to be the upper limit of the C group. (2) Add the standard deviation to the upper limit of the C's as found in step 1. This value may be taken as the upper limit of the B grades and all scores above this value marked as A's. (3) Subtract one-half the standard deviation from the mean. The value obtained may be taken as the lower limit of the C's. (4) Subtract the standard deviation from the lower limit of the C's. The result will be the lower limit of the D grades. All scores below this value may be marked F. A student of statistics will recognize that the procedure outlined is based on the assumption that scores falling within one-half unit of standard deviation above or below the mean—about 38 percent of all the scores—should be marked C; that scores falling between one-half and one and one-half units of standard deviation above the mean—about 24 percent of the total—should be marked B; that the corresponding group below the mean should be marked D; and that scores falling more than one and one-half units of standard deviation from the mean constitute the A's and F's. It must be remembered that these limits are set arbitrarily, that some other scheme could be used with equal claim to being "based on the normal curve," and that we have no assurance that our class is, in fact, normally distributed. But for an inexperienced teacher who has no other dependable basis for determining how to award letter grades, particularly in a large class, the scheme outlined will serve his purpose without doing any student grave injustice.

Following a procedure such as that outlined above has one grave defect, however. The performance of a particular class sets the standard for that class. If the average score of a class is low in comparison to what should be accomplished by a group of comparable ability, a relatively low score may receive a comparatively high grade. On the other hand, if the average performance of a class is high, then blind adherence to this plan might result in a student's receiving a D or F grade for a score that might be graded C or higher in a class making a lower average score.

Evidently, even in assigning marks, it is impossible to avoid having to make subjective judgments. A teacher should not try to avoid this responsibility. He should avoid making judgments based on insufficient evidence. With experience, he will be able to judge whether or not a particular class or section is above or below average and temper his marks accordingly. Use of standardized achievement tests will help the less experienced teacher to arrive at such judgments. If a class is unusually able, the teacher should not hesitate to give an unusual number of high marks. If the entire class is not doing as well as it should, there is no reason why the best student should be given an A grade if his work is really only average for the grade and course and stands out only by comparison with the less meritorious work of the other members of his particular class.

It is usually helpful, for both teacher and pupil, to set up descriptive criteria for each grade level. Thus, if a grade of A is defined as "Highly superior work; evidence of ability to work independently; essentially complete mastery of the subject matter of the course as evidenced by uniformly high grades on tests and ability to apply concepts and techniques to practical situations; consistent, thorough daily preparation," a teacher will not feel any obligation to mark a student A whose work has been erratic, high on some tests and low on others, or whose ability is great enough to permit him to pass tests with a reasonably high score but whose indolence and carelessness are evidenced by numerous small errors and by late or incomplete work.

In short, the marks assigned should clearly indicate, to a pupil and to others, the quality of the work done not only with respect to the rest of the class but also with reference to accepted objectives and standards for the work covered. A school mark should represent a considered judgment, by a competent judge, on the basis of extensive acquaintance, of the quality of the student's achievement. A conscientious teacher will be careful to make his marks mean just that.

SUMMARY

The evaluation phase of teaching mathematics should be an integral part of the total teaching program. The tests given, the marks awarded, should provide teacher, pupil, parents, prospective employers, and administrative officers with definite and specific information as to a student's achievement in terms of the objectives of the school and the course. The testing and evaluation program in mathematics should be a part of the total testing program of the school. The use of standardized intelligence, aptitude, and prognostic tests is becoming a part of the testing program of any efficiently run school. A teacher of mathematics needs to be sufficiently familiar with the theory and practice of educational measurement to be able to make full use of the information gained from such tests in working with his classes. In his own field, he will want to make use of standardized survey or achievement tests for comparative purposes and to help in evaluating the achievement of his classes. By so doing he will be able to gauge the effectiveness of his teaching and to formulate realistic standards of achievement for his pupils. He will use prognostic and diagnostic tests, both standardized and of his own devising, to insure the maximum effectiveness to his teaching. He will not be swayed by prejudice or slogan in choosing the type of test he employs but will use whichever type promises to achieve the end he has in view in giving it. On the other hand, he will recognize that even the best test is not an infallible measuring instrument, that his students are human beings, not statistical data, and that to ignore human considerations is neither "scientific" nor wise. In short, testing is an educative process. If it is properly done, it will contribute to the achievement of the aims of education. It is not an end in itself.

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Chapter 12

The Professional Preparation of Teachers

HENRY VAN DYKE once said that knowledge may be gained from books but that love of knowledge can be transmitted only by personal contact. President Garfield expressed the same idea in his oft-quoted remark that a boy on one end of a log and Mark Hopkins on the other constituted a school. In other words, the teacher is so important a factor in any teaching situation that all other considerations are of secondary importance by comparison.

The educative values that may be derived from a study of mathematics are not inherent in the subject matter; they are potentialities which may be realized by skillful teaching. The place of mathematics in secondary education depends, therefore, on the extent to which teachers of secondary mathematics can realize the potentialities inherent in their subject. As Butler and Wren point out, justification for the severe criticism of mathematics as an integral part of the secondary-school curriculum is to be found in the poor teaching, which leaves the impression that mathematics is a tool subject and nothing more.[1]

Unfortunately, the belief is widespread among laymen and school administrators alike that mathematics *is* merely a tool subject, the teaching of mathematics is simply a matter of drill on techniques, and, therefore, anyone who has had a few college hours of credit in the subject is qualified to teach it. To make a bad matter worse, the attitude of too many college professors of mathematics has been that expressed by the English mathematician, G. H. Hardy, who said, "The function of a mathematician is to do something, to prove new theorems, to add to mathematics, not to talk about what he or other mathematicians have done. . . . There is no scorn more profound, or on the whole more justifiable, than that of the men who make for the men who explain. Exposition, criticism, appreciation, is work for second-rate minds."[2]

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[1] Charles H. Butler and F. Lynwood Wren, *The Teaching of Secondary Mathematics*, New York, McGraw-Hill Book Co., 1941, p. 199.

Given mathematicians who, like Hardy, "hate teaching"[3] and, on the other hand, "teachers" who know little about mathematics and care less, it is not surprising that the teaching of mathematics in the elementary and secondary schools has been uninspiring and that large numbers of otherwise intelligent persons are willing to admit, sometimes even boast, that their ignorance of mathematics is complete and shameless. If a major portion of the blame for this unsatisfactory state of affairs, from the standpoint of mathematics, is to be charged to poor teaching and if the poor teaching is in turn to be charged to poor and inadequate preparation on the part of teachers of mathematics for their task, the obvious remedy is to require more adequate preparation of those who are now, or who plan to become, teachers of secondary mathematics.

The first and fundamental requisite in the preparation of a competent teacher of mathematics is thorough mastery of the subject. He must, as Young says, "have mastered it so well that he speaks with his own authority; only so can he hope to lead the pupil to the corresponding feeling of independent mastery. One can talk freely only about what he has really made his own. If he has not grasped the truth as distinguished from the form, if he feels bound to transmit unaltered to the pupil what he has received from some 'author'; he is bound hand and foot in the performance of his functions. Every day of a mathematics teacher's life demands that he vary unessentials to make essentials more evident."[4]

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[2] G. H. Hardy, *A Mathematician's Apology*, Cambridge, England, Cambridge University Press, 1941, p. 1.

[3] *Ibid.*, p. 89.

But knowledge of mathematics alone is not sufficient preparation for the task of teaching secondary mathematics. The major concern of the secondary school is not with subject matter but with persons. Hence, a competent teacher of secondary-school mathematics needs wide knowledge of people—of their aspirations, their needs, their strengths, and their shortcomings—so that he may use his knowledge of mathematics as a means of aiding his pupils to become competent citizens in a democratic society.

Finally, as was indicated earlier, since that love of knowledge from whence cometh wisdom is not taught but is gained from contact with an inspiring teacher, the good teacher of mathematics must be a personality in his own right. His preparation should be such, therefore, as will provide the requisite mastery of mathematics and an adequate basis for the understanding of people needed by a teacher, and in addition will develop in him those traits of character and personality requisite to one who is to guide and inspire youth.

Obviously, taking specific subject-matter courses will not guarantee that any individual will thereby become competent to teach secondary mathematics. A college charged with the task of preparing teachers can merely provide opportunity to learn the essential facts and to become acquainted with the ideals which should characterize the teacher. Whether an individual can transmute the learnings offered into the wisdom needed by a competent teacher depends on the personal resources which he possesses.

The specific subject-matter courses which a prospective teacher of secondary mathematics should take in fitting himself for his chosen vocation may, for purposes of discussion, be divided into general-education courses, professional courses in the theory and practice of teaching, and technical mathematics courses. There is, to be sure, no hard and fast line of demarcation between these various classes, but certain courses obviously make their major contribution to one or another particular area.

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[4] J. W. A. Young, *The Teaching of Mathematics in the Elementary and the Secondary School*, New York, Longmans, Green and Co., 1925, pp. 152–153.

GENERAL-EDUCATION COURSES

General-education courses are those which are a part of the education of every individual, whatever his profession or vocation. A prospective teacher of mathematics is, first of all, a human being with home, family, and community interests and relationships. He needs, in common with others, to possess the skills, understandings, aptitudes, and information necessary to maintain his physical, mental, and emotional health; to create and maintain a rich and satisfying home and family life; and to participate actively and effectively in the common life of his community.

A teacher of mathematics is also a citizen in a democratic society. He should, therefore, in common with all other citizens in such a society, possess an understanding of the major areas of human experience sufficient to enable him to participate intelligently and effectively in the political life of his community, state, and nation. He should also be able to manage successfully his own economic affairs and those public responsibilities which come to him. He should be able to observe, appraise, and react intelligently to new forces, movements, and trends in public affairs—economic, social, religious, and political. Likewise, he should be able to understand, enjoy, and make use of the products of the cultures of other lands and peoples.

But, as indicated above, courses and procedures designed to develop and impart these attitudes and learnings are not peculiar to the preparation of mathematics teachers. They are essential to the education of all men. They are what is properly termed general education. Hence, a competent teacher of mathematics needs a broad general education as a basis for his specialized preparation.

PROFESSIONAL EDUCATION COURSES

A competent teacher of secondary mathematics must have a clear understanding of the role of the school in our society. He needs to know what information, skills, habits of thought, and attitudes are desirable outcomes of the educational process. He needs to comprehend fully how instruction in mathematics contributes to the achievement of the purposes of education in a democratic society if he is to be able to translate effectively his knowledge of desirable goals and the resources of subject matter available into activities designed to achieve these accepted goals, and to be able to evaluate his work in light of progress made.

He must be skilled in the techniques and familiar with the literature of his profession. He should possess the essential teaching skills or techniques appropriate to his subject-matter field and be informed about current theories of child development, the psychology of learning, teaching and testing aids, techniques and procedures, curricular organization, and other professional knowledge pertinent to the task of teaching secondary mathematics.

It is, of course, the function of professional courses in education given as part of the teacher-training program to impart the skills, information, and attitudes listed above. However, courses in methods and in practice teaching which are an essential part of the professional preparation of every teacher should, for prospective teachers of secondary mathematics, emphasize methods which present mathematics as a mode for thinking—not as a body of manipulative techniques. Particularly in the field of geometry, prospective teachers should have knowledge of recent research and thought in teaching methods. Courses in teaching mathematics should include units on textbooks, workbooks, and reference materials, in which consideration is given to availability and uses and to problems of critical evaluation and selection. There should be similar units on audio-visual aids and materials which are commercially available or which may be made in the classroom, on mathematical recreations and clubs, on professional organizations and periodicals, and on opportunities for in-service growth and advanced study.

The work in methods and the courses in practice teaching should provide for consideration of problems such as content and organization of courses, blackboard techniques, testing and evaluation, homework, and the integration of work in mathematics with that of other courses. It is important that these courses be taught by instructors trained in mathematics as well as in teaching theory and with secondary-school experience and interests.

COURSES IN MATHEMATICS

The major purpose of specialized subject-matter courses in mathematics is, of course, to enable prospective teachers of mathematics to gain that scholarly competence which is essential to success as a teacher of the subject. Only if he has a broad and firm foundation in mathematics will a secondary-school teacher of mathematics be able to choose intelligently the ideas, concepts, and techniques which will be most significant to his students. It requires knowledge of mathematics far beyond the course being taught for a teacher to be able to suggest applications and modifications of textbook materials, to point out the why behind the processes presented, and to show the relationships between various areas of mathematics and the sciences where they find application. Only if he is a competent scholar in his field will he be able to teach mathematics as a way of thinking rather than as a body of manipulative techniques.

To gain scholarly competence in mathematics, prospective secondary-school teachers of the subject should begin their college work with the background of essentially complete mastery of the subject matter of elementary arithmetic, the equivalent of a year and a half of work in high-school algebra, and substantial knowledge of plane and solid geometry. Students lacking any part of this preliminary training should expect to remove the deficiency by the end of their first year of college work. In view of the large numbers of students who enter teacher-training institutions deficient in part or all of these preliminary requirements, it is likely that provision will have to be made by colleges to offer these courses in some form.

At the college level, the usual sequence through college algebra, trigonometry, analytic geometry, and the differential and integral calculus is almost universally required of prospective teachers of secondary mathematics. Where it is possible to offer it, there are good arguments in favor of a unified course covering the work through algebra, trigonometry, and analytic geometry. A recent study by Thacker and Read gives evidence of the reasonably close agreement among those best qualified to judge what constitutes desirable academic preparation for prospective teachers of secondary mathematics. They gathered the opinions of 263 high-school teachers of mathematics, 107 heads of college mathematics departments, and 83 heads of college departments of education relative to the college courses which were of most value in the preparation of secondary-school teachers of mathematics. The accompanying table presents the results of their survey.

Thacker and Read comment on the results they obtained as follows:

> The first point of interest is that for the ten top ranking courses, there is close agreement among the three different groups responding. Two exceptions are the listings given by college mathematics department heads to *Differential Calculus* and *Integral Calculus*. These courses were thought by them to be of greater importance than *High School Methods* and *Advanced College Algebra*. It is also interesting that both college mathematics and college education department heads rated *Educational Psychology* of somewhat greater importance than did high school mathematics teachers. College education department heads and high school mathematics teachers agree on the importance of *Advanced College Algebra*, whereas college mathematics department heads rated six other courses of more importance. *High School Methods* and *Educational Measurements*, both methodology courses, are included among these six.
>
> There was close agreement in regard to the ten top ranking courses. The next question is to determine if this agreement continues on down the list. The top twenty courses in rank were next studied and it appeared

College Courses of Value to High School Mathematics Teachers Ranked by Weighted Replies to the Questionnaire[5]

College Courses Listed	Rank Order			
	a	b	c	d
College Algebra	1	1	1	1
Plane Trigonometry	2	2	4	2
Methods in Teaching H.S. Mathematics	3	4	3	3
Solid Geometry	4	7	6	4
Educational Psychology	5	3	2	5
Supervised Teaching	6	6	5	6
Analytic Geometry	8	5	9	7
High School Methods	7	12	7	8
College Geometry	10	10	8	9
Advanced College Algebra	9	16*	10	10
Critical Review of Secondary Mathematics	11	14	11	11
Educational Measurements	13	15	12	12
Mental Hygiene	12	20*	18*	13
History of Mathematics	17	13	19*	14
Theory of Equations	20*	11	14	15
Survey of Mathematics	14	21*	13	16
Visual-Sensory Aids in Teaching	15	17	21	17
Philosophy of Education	19	19	15	18
Introduction to Education	21	18	16	19
School Curriculum	16	24	22	20
Vocational Guidance	18	22	27*	21
Differential Calculus	26	8*	20	22
Integral Calculus	30*	9*	23	23
Statistics	25	23	17*	24
Mathematics of Finance	22	26	25	25
School and Community Relations	24	27	24	26
Educational Sociology	23	29	29	27
History of Education	29	25	31	28
Spherical Trigonometry	27	32	28	29
Extracurricular Activities	28	30	32	30
Solid Analytical Geometry	33	28	26*	31
Projective Geometry	31	31	30	32
Supervision of Secondary Education	32	36	35	33
Comparative Education	34	33	38	34
Differential Equations	35	34	33	35
Secondary School Administration	37	35	34	36
Vector Analysis	36	38	37	37
Advanced Calculus	38	37	36	38

a—263 High School Mathematics Teachers
b—107 Heads of College Mathematics Departments
c—83 Heads of College Education Departments
d—453 Teachers and Heads of Departments (total of a, b, and c)
*—Ranks differing by five or more from that given by (d)

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[5] G. R. Thacker and C. B. Read, "Courses Desirable for Training Teachers of High School Mathematics," *School Science and Mathematics,* 49:611–619 (November, 1949).

that here the agreement was nearly as good as that noted for the top ten. High school mathematics teachers deemed *Vocational Guidance* of more importance than *Introduction to Education.* Heads of college mathematics departments again differed only in that they placed *Differential Calculus* and *Integral Calculus* above several other courses. In doing this they pushed *Survey of Mathematics* and *School Curriculum* down in the list. College education department heads placed *Statistics* and *Differential Calculus* ahead of *Visual-Sensory Aids* and *School Curriculum.* As a matter of fact they placed *Statistics* in a relatively higher spot than did the college mathematics department heads or the high school mathematics teachers. Both college education and mathematics department heads placed *Theory of Equations* somewhat higher than did high school teachers of mathematics.

At the bottom of the list of thirty-eight subjects, heads of college departments of education rated *Comparative Education* as least important. Heads of college mathematics departments gave this place to *Vector Analysis,* and high school mathematics teachers put *Advanced Calculus* in the lowest spot. The three groups placed essentially the same courses as the last five on the list. High school teachers agree with the composite ranking of the 453 replies. College education department heads placed *Supervision of Secondary Education* in this group instead of *Differential Equations.* College mathematics department heads placed *Supervision of Secondary Education* in place of *Comparative Education.*

If a course showed a discrepancy of five or greater from the rank given by the combined replies, this might indicate a marked difference of opinions. High school teachers of mathematics placed *Theory of Equations* in rank twenty, whereas the rank given by all teachers was fifteen for this course. High school teachers placed *Integral Calculus* at rank thirty in contrast to twenty-three for all teachers. College mathematics department heads placed *Advanced College Algebra* at sixteen in importance instead of ten, *Mental Hygiene* twenty instead of thirteen, *Survey of Mathematics* twenty-one instead of sixteen, and *Differential* and *Integral Calculus* at eight and nine instead of twenty-two and twenty-three. College education heads put *Mental Hygiene* and *History of Mathematics* at eighteen and nineteen instead of thirteen and fourteen, *Vocational Guidance* at twenty-seven instead of twenty-one, *Statistics* at seventeen instead of twenty-four, and *Solid Analytical Geometry* at twenty-six instead of thirty-one. Except for the calculus courses, these differences in rank never exceeded seven. [Italics in the original.][6]

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[6] *Ibid.*

Any summary of opinion remains just that. It may indicate a trend but it does not establish the validity of practices reported. Thus, as Thacker and Read show, the heads of college departments of mathematics reporting in their poll differed widely from the high-school teachers of mathematics and the heads of college departments of education on the value of courses in calculus. Yet it may be that the minority were more nearly correct in their appraisal than were the majority. It is difficult to see how a teacher of secondary mathematics can achieve the depth of insight into the fundamental concepts of mathematics that is essential if he is not acquainted with the characteristic concepts of differential and integral calculus. For example, a teacher without knowledge of the calculus will inevitably be handicapped in teaching theorems involving limits in geometry, in finding areas or volumes of irregular figures, or in appraising accurately the significance of theorems on incommensurate quantities. Morever, he simply cannot claim the scholarly competence in his field that a good teacher should have. Unfortunately, too often we think we are not using a course because we do not use the specific techniques taught therein. So it is with the calculus. A mathematics teacher will not, of course, be called upon to differentiate or to integrate in high-school classes in mathematics. But mastery of the techniques of differentiation and of integration, of the concepts of maxima and minima, of infinite series, and of indeterminate forms will give him insights into the problems of secondary-school algebra and geometry not available to him whose preparation is more limited.

It is probable, therefore, that mastery of the subject matter comprising the usual sequence through college algebra, trigonometry, analytic geometry, and the calculus is essential to preparation of competent teachers of secondary mathematics. In addition to these basic offerings, courses such as theory of equations, history of mathematics, college geometry, statistics, and mathematics of finance are highly desirable, if not absolutely essential. It may not be possible for a prospective teacher to take all of them at the undergraduate level but at least the three first named should be a part of the preparation of every teacher of secondary mathematics.

COURSES IN RELATED FIELDS

A competent teacher of secondary mathematics should have a broad knowledge of related fields. In particular, he should have sufficient acquaintance with the subject matter of physics to enable him to recognize applications of the mathematical techniques he teaches and to provide apt illustrations. In fact, rather extensive knowledge of physics is essential to full understanding of the power and significance of the concepts and techniques of mathematics.

The relationship between mathematics and the natural sciences or chemistry is not so close as that between mathematics and physics, but a competent teacher of secondary-school mathematics will need some acquaintance with the work of these fields so that he will know what use they make of mathematics and what materials drawn from their subject matter are pertinent to the work of his classes. Teacher-placement records, and surveys of common teaching combinations in the smaller schools, indicate that, other things being equal, a prospective teacher of secondary mathematics should minor in science.

Among related courses which will be found useful may be listed surveying and astronomy. Perhaps a course based on a text such as *Field Work in Mathematics* by Schuster and Bedford will be of more value than the more typical engineering courses in surveying. The subject matter of courses such as engineering drawing and descriptive geometry will be found useful to teachers of mathematics at any level.

There is less direct connection betwen courses in economics or other social sciences and mathematics than between mathematics and the physical sciences or engineering. However, there is coming to be increasing reliance on the techniques and concepts of statistics in studying the problems of these areas. Moreover, since a teacher of mathematics cannot exist other than as a member of various social groups, both as a person and as a teacher of mathematics, he should have sufficiently broad acquaintance with the subject matter of the social sciences in general to function effectively as a citizen and to recognize the differing types of mathematical treatment required by typical problems in this area and, again, to be able to utilize pertinent materials from this area in his teaching.

PREPARATION AT THE GRADUATE LEVEL

A teacher of secondary mathematics will require a minimum of four years of college work and should expect to complete a fifth year of academic preparation as soon as practicable, if he is to achieve the degree of scholarly competence in mathematics and in related fields that is essential to effective teaching. His academic training, both in his own field and in related areas, should aim at breadth, yet maintain sufficient depth to insure a requisite degree of scholarly competence. At the graduate level, particularly, his preparation should differ markedly from that of a prospective research worker in pure mathematics. The research worker needs to master thoroughly the relatively restricted area in which he is working to the end that he can add to the store of knowledge in that area. The teacher of secondary mathematics is more concerned with the interrelationships among the various areas of mathematics and with their significance from an educational point of view.

The fifth year of academic work taken in preparation for teaching secondary mathematics may well include some courses designed to impart further mastery of the subject matter of mathematics and to acquaint teachers with typical concepts and techniques of advanced mathematics. Courses such as advanced calculus, differential equations, projective geometry, higher algebra, solid analytical geometry, theory of probability, mathematical statistics, and additional work in analytical mechanics or electronics in the field of physics would be desirable.

However, at least half of the work of the fifth year should be devoted to courses having to do with the teaching of mathematics.

This portion of the fifth-year program may include additional courses in psychology and pertinent courses in education. It should include work in teaching and supervision of mathematics classes, including a critical study of the literature of the field. In this or related studies, a teacher should become familiar with visual-aid techniques and resources, testing and remedial techniques applicable to secondary mathematics, and problems of planning and administering the secondary-school curriculum and the function of mathematics in that curriculum.

Ideally, the program of the fifth year should provide experience in doing independent research on teaching problems and in making scholarly reports of the outcomes and findings of such research.

THE PROBLEM OF THE NONMATHEMATICAL STUDENT

One of the most pressing and important problems facing teachers of secondary mathematics today is that presented by the nonmathematical student—an individual with no aptitude for, liking for, or desire to learn mathematics. The problem can be solved only by a teacher who is familiar with and sympathetic to the needs of this large class of students. A successful teacher must be able and willing to utilize the subject matter of mathematics to instruct nonmathematical students, rather than attempting merely to teach mathematics to them. No specific course will serve to prepare teachers for this work. The preparation needed is a matter of attitude on the part of the prospective teacher—a realization that people are more important than subject matter. To do a good job of teaching the typical general mathematics course requires far more of a teacher in the way of mathematical knowledge, of insight into human nature, and of dedication to the ultimate aims of education than anyone realizes if he has not come to grips with the problem.

Possibly the most that can be done in the matter of preparing teachers competent and willing to attempt to master the complex and difficult techniques of teaching nonmathematical students by means of mathematics is to emphasize continually what Nowlan has said about teaching college mathematics: "There are two main aims in the teaching of college mathematics. . . . These, in order of importance are: (1) training in precise thinking and a grasp of basic principles. (2) The acquisition of information and a mastery of certain basic skills."[7] Nowlan concludes, "It appears, however, that, in general, a disproportionate stress has been placed on the last objective to the almost total neglect of the first."[8] If prospective teachers of mathematics can be brought to see that the first objective can be realized with classes of nonmathematically inclined students and that they can contribute much to mathematics, to education, and to their students by working persistently in this area, perhaps the future will see these difficult classes assigned only to the best-prepared and most competent teachers, instead of being "farmed out" to anyone having a vacant period at the hour at which they happen to be scheduled. And when competent, well-prepared teachers of mathematics really try to do something about the problem presented by nonmathematically inclined students, they may be surprised to discover that mathematics can be better served by attempting to teach Johnny by means of mathematics than by attempting to teach mathematics to Johnny.

IN-SERVICE ACTIVITIES IN TEACHER EDUCATION

It is not possible to anticipate in college mathematics classes and in professional education classes everything one will need to know in order to solve all the problems which will confront him as a teacher of secondary mathematics. In the first place, an inexperienced teacher lacks a background of experience essential to a full appreciation of the problems he will encounter. Secondly, conditions change, bringing new problems in the wake of each change. Therefore, a competent, professionally conscientious teacher will need, and want, to participate regularly in in-service activities designed to broaden his professional preparation and to keep him abreast of the times. In teaching, as in any other profession, change is rapid and the rate at which currently accepted concepts and practices become obsolete is so great that continuous study and effort are required of each practitioner who would remain professionally alive.

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[7] F. S. Nowlan, "Objectives in the Teaching of College Mathematics," *The American Mathematical Monthly*, 57:74 (February, 1950).

[8] *Ibid.*

Study leaves and attendance at summer sessions of colleges and universities are obvious ways in which a teacher may continue to broaden his preparation after beginning the active practice of his profession. But not everyone can take time off for additional study and, eventually, there comes a time when the mere piling up of additional credit loses attraction.

Extensive reading of professional books and journals is another excellent means of keeping informed about current developments in the field of one's interest. The preparation of papers to be read at professional meetings or for publication in the journals is a potent means of stimulating purposeful reading and thinking about current issues of concern to teachers and to citizens in general. A teacher of secondary mathematics may not be able to create mathematics in the sense that research mathematicians such as Hardy do but he can be equally creative in his own field. The outcome of creative work in his field is the discovery and development of new mathematicians and is certainly more important than the discovery or creation of new mathematics. Hence, so that he may keep professionally alive, a teacher in service should be continually reading, studying, thinking, and experimenting with new methods, new ways of doing things.

Membership in and attendance at meetings of various educational and mathematical organizations will afford opportunities to broaden acquaintances within the profession, to exchange experiences, and to gain new ideas. The professional organizations to which a secondary-school teacher of mathematics may well belong are of two types: those to which all teachers may belong, such as the national, state, and local teachers' associations, and the associations of teachers of mathematics alone. There are three organizations for teachers of mathematics that are national in scope. The oldest, the Mathematical Society of America, is primarily for research mathematicians. However, a secondary teacher who is interested in keeping alive an interest in pure mathematics may find the stimulation offered by the meetings and publications of the Society helpful.

The American Mathematical Association is an organization of college and secondary teachers of mathematics. It gives more attention to the problems of teaching mathematics than does the Mathematical Society although the papers presented at its meetings or published in its journal, *The American Mathematical Monthly*, are primarily of interest to teachers at the college level and will not often have much bearing on the high-school teacher's problems. The Association will, however, afford opportunity to keep alive interest in pure mathematics and will probably serve this purpose better than will the Mathematical Society.

The National Council of Teachers of Mathematics is an organization of secondary-school mathematics teachers and of college teachers concerned with the preparation of secondary-school teachers of mathematics. It publishes *The Mathematics Teacher*, a journal devoted chiefly to articles of interest to teachers of secondary mathematics. There is usually a spring meeting of the National Council in April, a summer meeting in August, and a winter meeting during the Christmas holidays.

The Central Association of Science and Mathematics Teachers, Inc., publishes the magazine *School Science and Mathematics*. This organization and its magazine are, as their titles indicate, of interest to teachers of secondary-school science and mathematics. There are several other organizations of teachers of mathematics which are state-wide or regional in character. Then there are organizations for those interested in particular phases of mathematics such as the American Statistical Association, the Actuarial Society of America, and the American Institute of Actuaries. The American Association for the Advancement of Science has a mathematics section and also a section on education so that teachers of secondary mathematics may find something of interest in its program and its publications.

A valuable type of in-service training for teachers is that afforded by summer employment in industries which are characterized by extensive applications of mathematical concepts. A teacher who spends a summer working in the engineering department of a construction firm or the statistical department of a great corporation will gain insights into the applications of mathematics not available to one without such experience. Since, however, such opportunities are not widely available to all teachers, a satisfactory substitute may be found in the mathematics workshops currently being sponsored by various colleges and organizations of teachers of mathematics during the summer vacation.

It is evident, therefore, that there is no dearth of opportunities for a teacher of secondary mathematics to extend his preparation for teaching either through further academic work or through some form of in-service training. The only thing needed is a desire to grow professionally.

THE TASK OF THE TEACHER

A competent teacher of secondary mathematics will always keep in mind that the end product of the educative process in a democratic society should be an independent learner competent to assume the responsibilities of citizenship in such a society. An individual becomes competent to order his own life and to function effectively as a citizen in a democratic society when, and to the extent that, he is able to substitute a reasoned approach to the problems of life for the haphazard and often emotional approach of an immature, or unlearned, or uncritical, unthinking individual. A competent teacher of mathematics will recognize, therefore, that his first concern is for the student and that the subject matter taught is but the means of aiding his students to achieve competence to cope with the problems of life. Furthermore, because he has adequate breadth of scholarship in mathematics and in professional courses in education, a teacher will recognize that mathematics is, basically, a mode of thinking, a vehicle for working with and conveying ideas rather than a mere collection of facts, skills, and techniques. He will, therefore, recognize that instruction in mathematics is an essential part of the educational experience of every individual and be able to demonstrate to parents, school administrators, and laymen in general that mathematics does function in the everyday life of the individual; to make his classes worth while and stimulating; and to lead his students to enjoy mathematics.

Through much of the literature on the problems of teaching mathematics, as a consequence of the vehement attacks on its educational values, runs a thread of fear for the future of mathematics. There would seem to be no real need to worry about that problem. The tremendous growth of mathematical theory in the past half-century is sufficient evidence of its vitality as one of the great "major trunk lines of human endeavor." The tremendous reliance on mathematical techniques in scientific research and in industry is ample assurance that there will always be a demand for mathematically trained specialists. Moreover, a real research mathematician, like an artist or a musician, will persist in following his bent despite all discouragements. There will always be, also, those who recognize the necessity of adequate foundation in mathematics if they are to achieve their goals in engineering, science, or other vocational or intellectual activities where mathematical training is prerequisite. Hence, there is little danger that the study of mathematics will disappear completely from our schools.

The real danger is that too many students are failing to get an acquaintance with mathematics sufficiently broad to enable them to realize fully their own potentialities. For many, perhaps for a great majority of students in the secondary schools who have as yet no particular preference for any vocation or intellectual objective and whose interest—or dislike—for mathematics is only casual, the road to genuine interest in and liking for mathematics lies through an inspiring teacher. Many a student has discovered an aptitude for mathematics because he first took a liking to the teacher. Therefore, the primary responsibility of a teacher of secondary mathematics is to make his classes interesting, challenging, and worth while. The potency of medicine is not decreased by sugar-coating a pill, and a course in high-school algebra or geometry can be mathematically rigorous and still be so interesting that students will look forward to that class with eager anticipation each day. The key is the teacher. If mathematics is dull, dry, and lifeless, it is the fault of the teacher. If a mathematics class is stimulating, alive, and enjoyable, it is because the teacher makes it so.

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